

POSITIVE REPRESENTATIONS OF GENERAL COMMUTATION RELATIONS ALLOWING WICK ORDERING

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ABSTRACT. We consider the problem of representing in Hilbert space commutation relations of the form

$$a_i a_j^* = \delta_{ij} \mathbf{1} + \sum_{k\ell} T_{ij}^{k\ell} a_\ell^* a_k \quad ,$$

where the $T_{ij}^{k\ell}$ are essentially arbitrary scalar coefficients. Examples comprise the q -canonical commutation relations introduced by Greenberg, Bozejko, and Speicher, and the twisted canonical (anti-)commutation relations studied by Pusz and Woronowicz, as well as the quantum group $S_\nu U(2)$. Using these relations, any polynomial in the generators a_i and their adjoints can uniquely be written in “Wick ordered form” in which all starred generators are to the left of all unstarred ones. In this general framework we define the Fock representation, as well as coherent representations. We develop criteria for the natural scalar product in the associated representation spaces to be positive definite, and for the relations to have representations by bounded operators in a Hilbert space. We characterize the relations between the generators a_i (not involving a_i^*) which are compatible with the basic relations. The relations may also be interpreted as defining a non-commutative differential calculus. For generic coefficients $T_{ij}^{k\ell}$, however, all differential forms of degree 2 and higher vanish. We exhibit conditions for this not to be the case, and relate them to the ideal structure of the Wick algebra, and conditions of positivity. We show that the differential calculus is compatible with the involution iff the coefficients T define a representation of the braid group. This condition is also shown to imply improved bounds for the positivity of the Fock representation. Finally, we study the KMS states of the group of gauge transformations defined by $a_j \mapsto \exp(it)a_j$.

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0. INTRODUCTION

0.1. Main results.

This paper is about representations in Hilbert space of a wide class of involutive algebras, which are obtained from – often only finitely many – generators a_i with the relations

$$a_i a_j^* = \delta_{ij} \mathbf{1} + \sum_{k\ell} T_{ij}^{k\ell} a_\ell^* a_k \quad , \quad (0.1.1)$$

where the $T_{ij}^{k\ell}$ are complex coefficients constrained only by a hermiticity condition so that the above relation respects the involution. The simplest case of such an algebra are certain deformations of the unit disk [NN]. Extensions to relations of the form $aa^* = f(a^*a)$ for a single generator, with more general function f have also been studied [Das], and have been shown to be analyzed completely in terms of a weighted shift [KL]. In this paper we look at generalizations to more than one generator, in which no simple weighted shift representation is possible. Our paper does subsume many examples found in the literature, however, and unifies, as well as substantially extends previously known results.

In a basis free language the “structure constants” $T_{ij}^{k\ell}$ determine an operator $\tilde{T} : \mathcal{H}^\dagger \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}^\dagger$, where \mathcal{H} is a Hilbert space with a basis $\{e_i\}$ labelled as the generators, and \mathcal{H}^\dagger is the conjugate of \mathcal{H} . The relations

$$f^\dagger \otimes g = \langle f, g \rangle \mathbf{1} + \tilde{T}(f^\dagger \otimes g) \quad , \quad f^\dagger \in \mathcal{H}^\dagger, g \in \mathcal{H} \quad (0.1.2)$$

then define an ideal in the algebra of all tensors over \mathcal{H} and \mathcal{H}^\dagger with tensor multiplication as product. The quotient of the tensor algebra by this ideal is then an abstract involutive algebra, which we denote by $\mathcal{W}(T)$. A fundamental algebraic property of $\mathcal{W}(T)$ is that by using (0.1.1) every expression can be rearranged uniquely in “Wick ordered form”, such that, in every monomial, all starred generators are to the left of all unstarred ones. To emphasize this property, $\mathcal{W}(T)$ will be called a “Wick algebra”. We are mainly interested in “positive representations” of $\mathcal{W}(T)$, i.e. representations of the a_i as operators on a Hilbert space such that a_i^* is a restriction of the operator adjoint of a_i . Moreover, we would like to characterize the solutions of (0.1.1) by bounded operators.

For any choice of $T_{ij}^{k\ell}$, $\mathcal{W}(T)$ has a unique representation constructed from a cyclic vector Ω with the property that $a_i \Omega = 0$ for all generators a_i . This representation, called the Fock representation, carries a natural hermitian scalar product, which, however, often fails to be positive semi-definite. One of the main problems addressed in this paper is to find criteria on $T_{ij}^{k\ell}$ implying the positivity of the Fock representation. We generalize the Fock representation to the so-called coherent representations, the representations generated from a cyclic vector Ω such that $a_i \Omega = \varphi_i \Omega$, with $\varphi_i \in \mathbb{C}$. We will demonstrate the usefulness of this concept for constructing representations.

We have shown in earlier work [JSW1] that, provided the coefficients $T_{ij}^{k\ell}$ are sufficiently small, the Fock representation is positive and bounded, and is, moreover, the universal

bounded solution of the relations (i.e. every bounded solution is in a quotient of the C^* -algebra generated by the a_i in the Fock representation). Moreover, the universal C^* -algebra is independent of (small) T , and is isomorphic, as a C^* -algebra, to an extension of the Cuntz algebra [Cu1] by the compact operators. In [JSW1] we did not give explicit bounds on T . It might appear that such bounds were best given in terms of the norm of the operator \tilde{T} appearing naturally in (0.1.2), and the definition of the algebra $\mathcal{W}(T)$. It is one of the main conclusions of this paper that this is misleading, and that another operator, which we will simply denote by T , plays a fundamental rôle. T is a partial adjoint of \tilde{T} with respect to only one tensor factor. Explicitly, we set $T : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ with

$$Te_i \otimes e_\ell = \sum T_{ij}^{k\ell} e_j \otimes e_k \quad . \quad (0.1.3)$$

Our new, explicit criterion for the isomorphism of the universal bounded representation of $\mathcal{W}(T)$ with the extended Cuntz algebra (Theorem 2.4.4 below) becomes $\|T\| < \sqrt{2}-1$, with further refinements if $T \geq 0$ or $T \leq 0$. Our main results regarding the Fock representation are summarized in the following Theorem. The third item was shown recently by Bożejko and Speicher [BS2] using techniques rather different from those employed in this paper. In its statement, $T_1 = T \otimes \mathbb{I}$, and $T_2 = \mathbb{I} \otimes T$, acting on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$.

Theorem 0.1.1. *Let $T_{ij}^{k\ell}$ denote the coefficients defining a Wick algebra on finitely many generators, and let T be the operator defined in (0.1.3). Then the Fock representation of the Wick algebra $\mathcal{W}(T)$ has positive semi-definite scalar product, if any one of the following conditions holds:*

(Theorem 2.3.2) $\|T\| \leq 1/2$.

(Theorem 2.5.1) $T \geq 0$.

(Theorem 2.6.3) T satisfies the braid relation $T_1 T_2 T_1 = T_2 T_1 T_2$, and $\|T\| \leq 1$.

Moreover, all three bounds (1/2, 0, and 1) are best possible.

In the definition of Wick algebras, we have imposed no relations between the generators a_i . The idea of Wick ordering, however, makes sense also in the presence of such relations, provided these relations are compatible with the coefficients $T_{ij}^{k\ell}$ in a suitable sense. Ideals in the tensor algebra over \mathcal{H} satisfying this condition will be called *Wick ideals* (see Section 3 for details). An interesting observation is that in some cases such an ideal is annihilated by every bounded representation.

As a case study of the interplay between the structure of Wick ideals and the positivity of representations, we consider three structures defined by Woronowicz and Pusz, namely their twisted canonical commutation relations ([PW], see (1.2.3)), the twisted canonical anti-commutation relations ([Pus], see (1.2.4)), and the quantum group $S_\nu U(2)$ ([Wo1], see (1.2.5)). In each case the original definition uses relations of the form (0.1.1), as well as relations between the a_i alone. In this paper we consider, in each case, the Wick algebra defined by retaining only the Wick algebra relations, but discarding any further relations.

Theorem 0.1.2. *Consider an irreducible representation π of the Wick algebra $\mathcal{W}(T)$ by bounded operators on a Hilbert space \mathcal{H} , where T is given by one of the following:*

- (1) *the twisted canonical commutation relations (cf. (1.2.3); Theorem 3.2.1)*
- (2) *the twisted canonical anti-commutation relations (cf. (1.2.4); Theorem 3.2.3)*
- (3) $S_\nu U(2)$ (cf. (1.2.5); Proposition 1.2.4)

Then π is coherent. More specifically there are a cyclic vector $\Omega \in \mathcal{H}$, $k \in I$, and $z \in \mathbb{C}$ such that $a_k \Omega = z \Omega$, and $a_i \Omega = 0$ for $i \neq k$.

In the case of the twisted canonical commutation relations (resp. $S_\nu U(2)$) this result implies that the additional relations automatically follow from the boundedness (resp. positivity) of the representation; in the case of the twisted canonical anti-commutation relations we also find representations in which these relations are *not* satisfied. In all cases, the parameters k and z defining the coherent representation provide an efficient parametrization of the equivalence classes of irreducible representations, and we obtain a neat summary of the classification work done in [PW,Pus,Mei], correcting, in passing, an omission of some cases in the classification of [PW].

Relations of the form (0.1.1) appear naturally as the commutation rules between coordinates and partial derivatives in *non-commutative differential geometry*. We indicate the connections, and point out a characterization of the braid relation for T in terms of differential geometry (Theorem 4.3.2). Finally, we study the automorphism group acting on the generators as $a_j \mapsto e^{-it} a_j$, and its associated KMS-states.

0.2. General background and related work.

Many of the structures that have been studied in recent years as deformations of “classical” objects, such as quantum groups [Wo2,FRT], and quantum planes [Man,WZ], as well as various deformations of the canonical commutation relations [BS1,PW] are defined in terms of algebraic relations between the generators of an algebra. Consequently, much of the work on these structures has been in the purely algebraic category. An aspect that has often been neglected is the question under what conditions a given set of relations can be realized by (possibly unbounded) operators in a Hilbert space. This is unfortunate, both from a physical and a mathematical point of view.

From the physical point of view, it is desirable to be able to interpret the algebra under consideration as the algebra of observables of a physical quantum system. Then the statistical interpretation of quantum theory gives special importance to the positive elements of the algebra, since only these can describe probabilities. The positive elements are in turn defined as those of the form X^*X , in terms of an involution $X \mapsto X^*$. Thus observable algebras are always involutive algebras with a generating positive cone. This in turn implies that the algebra has a faithful representation by Hilbert space operators. We do not claim, of course, that these are the only algebras that may have “physical” significance. A good example to the contrary is the superselection structure in low dimensional quantum

field theory, often described in terms of the representation theory of some quantum group [Reh], which itself is not interpreted as an observable algebra. Nevertheless, we feel that the observable interpretation of quantum theory is too fundamental to be ignored lightly in the development of new algebraic structures for quantum theory.

On the mathematical side the relevance of Hilbert space representations (“positive definite”, or “positive energy” representations for emphasis [KR,Jor,BELGJ]) is obvious from the example of Lie algebra theory, where a rich structure unfolds even for finite dimensional algebras. Picking a setting with representations realized in Hilbert space demands positivity, and in a related way positivity plays a basic rôle in [Jo1]. The algebras we consider in this paper are all infinite dimensional (although typically with a finite number of generators), so an additional aspect is the necessity of topological concepts to tame the various infinite sums arising in the theory. An example from quantum group theory for the need for topological completions is the identification of the tensor product of two function algebras with the algebra of functions on the Cartesian products. This is usually taken to motivate the definition of the coproduct, but generally fails for the *algebraic* tensor product of algebras of continuous functions on non-discrete spaces. The topological notions needed for such purposes in the non-commutative situation are often imported most naturally from a collection of Hilbert space representations.

In Sections 2 and 3 we show that there is a surprising interplay between the purely algebraic side of the theory and the constraints imposed by demanding a Hilbert space representation. We show that often there will be additional relations, which do not follow *algebraically* from the given ones, which are nevertheless valid in every Hilbert space representation. This phenomenon is part of a general theory of C*-algebras defined in terms of generators and relations, which has been studied recently by several authors [KW,Bla,SW,Phi].

The choice of the relations (0.1.1) was made since several examples of such structures are currently under investigation in the literature. Our initial motivation was to understand the relationship between two different deformations of the canonical commutation relations, namely the “ q -deformed” commutation relations [BS1,Gre,JW], and the “twisted” canonical commutation relations [PW,Pus]. The latter are given as two sets of relations, one set of the form (0.1.1), and another set involving only the generators a_i . This is in contrast to the situation for the q -deformed relations, where any such additional relations lead to inconsistency. In order to understand the relationship between these structures we will treat both cases from the point of view of (0.1.1) alone. Admissible relations between the a_i then correspond precisely to Wick ideals studied in Section 3.

The Wick order viewpoint is also a natural ingredient in the Wess-Zumino [WZ], and Woronowicz [Wo3], approach to non-commutative differential geometry, and we shall take up this point in Section 4 below; see also Baez [Bae] where the classical Poincaré lemma from de Rham theory is proved in the quantum theoretic setting. Here the adjoints a_j^* of the generators a_j are interpreted as partial derivatives on the algebra of “functions”

generated by the a_j alone. This is motivated by the Segal-Bargmann [Fol] formalism for the complex formulation of the Schrödinger representation. The fundamental idea in [Bar] was to pick a positive definite inner product on the functions of several complex variables (z_1, \dots, z_d) such that the multiplication operator $f(z) \mapsto z_j f(z)$ becomes adjoint to $\partial/\partial z_j$, in the sense

$$\left\langle z_j f(z), h(z) \right\rangle = \left\langle f(z), \frac{\partial}{\partial z_j} h(z) \right\rangle \quad (0.2.1)$$

for functions $f(z), h(z)$ in the corresponding Hilbert space, or Bargmann space. Specifically, Bargmann finds his (now familiar) Gaussian Hilbert space as a solution $\langle \cdot, \cdot \rangle$ to the ansatz (0.2.1). Our viewpoint here is that for the general Wick relation (0.1.1), the requirement of hermiticity, i.e. that representations take the involution in the abstract Wick algebra into the operator theoretic adjoint, serves as a generalization of (0.2.1). It turns out that this approach is equivalent to defining an algebra of differential forms, with an additional commutation rule transforming expressions of the form “ $x_i dx_j$ ” into “ $dx_k x_\ell$ ”. For generic coefficients, however, all differential forms of degree 2 and higher vanish. In the presence of Wick ideals non-trivial higher order forms can exist. In the classical case the anticommutativity of the differentials implies that any form of degree higher than the number of generators vanishes. This is not true in the non-commutative setup discussed here, and we will give an example of a calculus with forms of all orders, using recent results from the theory of spin chains [FNW].

For further examples of Wick algebras, and more details on those mentioned here, we refer to Section 1.2.

The term “*Wick ordering*” for the process of ordering all creation operators to the left of all annihilation operators is standard usage in physics for the canonical (anti-) commutation relations. We therefore use the term “*Wick algebra*” for an algebra obtained from relations of the form (0.1.1). The term has been used before by Slowikowski [Slo] in a much more special (commutative) context. The terms “*coherent representation*” for the “highest weight” representations with a cyclic vector Ω satisfying $a_i \Omega = \alpha_i \Omega$ with $\alpha_i \in \mathbb{C}$, and “*Fock representation*” for the special case $\alpha_i \equiv 0$ are also in agreement with standard terminology for the canonical (anti-) commutation relations. We give an algebraic definition of these representations, and the canonical hermitian forms on the respective representation spaces, for any choice of coefficients $T_{ij}^{k\ell}$, and eigenvalues α_i . Our main problem, namely deciding the positivity of this hermitian form is reminiscent of the positivity question which is central to the study of the coherent representation of the *Virasoro algebras* \mathcal{V} (see [KR]); in the simplest case these representations are labeled by two parameters (c, λ) where c is the central charge, and λ is the spectral value for the Cartan generator of \mathcal{V} . The coherent representation is then defined by the highest weight method and the representation space is the corresponding Verma module (see [GW, KR]). The explicit range of the parameters (c, λ) where the corresponding inner product $\langle \cdot, \cdot \rangle_{c, \lambda}$ is positive semi-definite is known, see [KR] and [GW] for details. In a different context, a similar positivity condition dictates the values of the Jones index [Jo1, Jo2].

As Theorem 0.1.1 shows, Wick algebras with small $\|T\|$ can be understood in a rather general way. The universal bounded representation exists by Theorem 2.4.4, and is isomorphic to the Cuntz-Toeplitz algebra, which has only one proper closed two-sided ideal. Hence no relations between the generators, in addition to (0.1.1), can be imposed. When $\|T\|$ exceeds 1 this isomorphism breaks down. As a simple example in which additional relations are not only consistent with (0.1.1), but are forced by it in conjunction with the positivity requirement, we present the C^* -algebra of Woronowicz' quantum group $S_\nu U(2)$. More generally, the possibility of further relations, conveniently expressed as the existence of a non-trivial "Wick ideal" implies that -1 is an eigenvalue of T , and the Fock scalar product becomes partly degenerate. These Wick ideals are a typical feature of the twisted canonical commutation relations [PW,Pus], and also play a fundamental rôle in setting up a differential calculus [WZ] (see below). Typically, Wick ideals satisfy a version of the relations (0.1.1) without constant term. This allows us to conclude that the Wick ideal in twisted canonical commutation relations [PW] is automatically annihilated in every bounded representation, and to describe the structure of representations of the twisted canonical anti-commutation relations not assuming relations between the a_i . The "untwisted" case is the theory of Clifford algebras, and was treated from this point of view in [JW].

Finally, we briefly consider the KMS states associated with the one parameter automorphism group α acting on the generators as $\alpha_t(a_j) = e^{-it}a_j$. In Fock space this automorphism is generated by the number operator, and we connect the growth of the n -particle subspaces to the existence of KMS states.

Explicit computations with Wick relations can be quite painful, since each application of the rule (0.1.1) may multiply the number of terms involved. For the present paper the largest computations of this sort were necessary to verify the Wick ideal property in Theorem 3.2.4. We therefore developed a little MATHEMATICA [Mat] package to perform such computations. It takes the commutation rules as a parameter, and computes Wick ordered forms, as well as Fock and coherent functionals. We have made it available by anonymous ftp from "nostromo.physik.Uni-Osnabrueck.de" in the directory "pub/Qrelations/Wick".

1. BASIC DEFINITIONS AND EXAMPLES

1.1. Hermitian Wick algebras.

In this section we supply the general facts on the algebra $\mathcal{W}(T)$ based on the relation (0.1.1). This will be the initial axiom. We found it convenient to write the generators not as indexed quantities " a_i ", with redundant letters " a " cluttering every computation, but to denote the generators directly by " i ". Moreover, we changed the position of the stars relative to (0.1.1). The connection with the notation of (0.1.1) will be made explicit once more in (1.1.6) below.

Definition 1.1.1. Let I be a set, and let $T_{ij}^{k\ell} \in \mathbb{C}$ for $i, j, k, \ell \in I$ such that for each pair i, j only finitely many $T_{ij}^{k\ell} \neq 0$. Then the **Wick algebra** on the coefficients T , denoted by $\mathcal{W}(T)$, is the algebra of polynomials (possibly with constant term) in the symbols i, i^\dagger for $i \in I$, divided by the relation

$$i^\dagger j = \delta_{ij} \mathbb{1} + \sum_{k, \ell \in I} T_{ij}^{k\ell} \ell k^\dagger, \quad (1.1.1)$$

A **Hermitian Wick algebra** is a Wick algebra with involution “ \dagger ” extending the map $i \mapsto i^\dagger$. The existence of this involution is equivalent to the condition

$$T_{ji}^{\ell k} = \overline{T_{ij}^{k\ell}}. \quad (1.1.2)$$

The distinguishing feature of the relations (1.1.1) is that they allow us to order any polynomial in the generators i, i^\dagger such that in every monomial all i are to the left of all i^\dagger . The result of this transformation is called the *Wick ordered form* of the given polynomial. It is unique, i.e. independent of the order in which the rules are applied to different parts of the expression. The proof given in the following Lemma is based on a simple graph theoretical principle known as the Diamond Lemma [Ber].

Lemma 1.1.2. Let $\mathcal{W}(T)$ be a Wick algebra. Then the Wick ordered monomials form a basis of $\mathcal{W}(T)$.

Proof. In order to apply the theory of algebraic reduction presented in [Ber] we have to verify two things: firstly, the successive application of Wick ordering substitutions must terminate, which is obvious. Secondly, the so-called diamond condition has to be satisfied, i.e. whenever there are two different possibilities for applying the substitution rules to the same expression, there must be further substitutions on the respective results leading to the same final result. In a Wick algebra, a monomial in which two different substitutions are possible is of the form $X i^\dagger j Y k^\dagger \ell Z$, where i, j, k, ℓ are generators, and X, Y, Z arbitrary (possibly empty) monomials. Since the two possible substitutions in this situation commute, the diamond condition is also obviously satisfied. \square

By \mathcal{H} we will denote the free vector space over I , i.e. the set of finite linear combinations $f = \sum_{i \in I} f_i i$. Then $f^\dagger = \sum_{i \in I} \overline{f_i} i^\dagger$ is considered in a natural way as an element of the complex conjugate space \mathcal{H}^\dagger of \mathcal{H} . The canonical inner product of \mathcal{H} , denoted by

$$\langle f, g \rangle = \sum_i \overline{f_i} g_i \quad (1.1.3)$$

makes \mathcal{H} into a pre-Hilbert space. The free algebra over the generators i, i^\dagger is the same as the tensor algebra

$$\mathcal{T}(\mathcal{H}^\dagger, \mathcal{H}) = \mathbb{C}\mathbb{1} \oplus \mathcal{H} \oplus \mathcal{H}^\dagger \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H}^\dagger) \oplus (\mathcal{H}^\dagger \otimes \mathcal{H}) \oplus (\mathcal{H}^\dagger \otimes \mathcal{H}^\dagger) \oplus \dots \quad (1.1.4)$$

over \mathcal{H} and \mathcal{H}^\dagger , i.e. the direct sum of all tensor products with factors \mathcal{H} or \mathcal{H}^\dagger , with tensor product as multiplication. The empty tensor product is identified with \mathbb{C} , and this summand of $\mathcal{T}(\mathcal{H}^\dagger, \mathcal{H})$ contains the unit $\mathbb{1}$ of the algebra. We will often omit the tensor product signs, using them only for punctuation improving the legibility of a formula. The Wick algebra $\mathcal{W}(T)$ is then the quotient of $\mathcal{T}(\mathcal{H}^\dagger, \mathcal{H})$ by the ideal in $\mathcal{T}(\mathcal{H}^\dagger, \mathcal{H})$, generated by $f^\dagger \otimes g - \langle f, g \rangle \mathbb{1} - \tilde{T}(f^\dagger \otimes g)$, where \tilde{T} denotes the linear operator

$$\tilde{T} : \mathcal{H}^\dagger \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}^\dagger \quad : f^\dagger \otimes g \mapsto \sum_{i,j,k,\ell} T_{ij}^{k\ell} \overline{f_i} g_j \ell \otimes k^\dagger \quad . \quad (1.1.5)$$

Note that using the operator \tilde{T} we can state the relations (1.1.1) in basis-free form. This will be useful for extending the above structures to suitable completions of \mathcal{H} and $\mathcal{W}(T)$ (compare Example 1.2.8).

We are mainly interested in representations of a Wick algebra by linear operators on a complex vector space \mathcal{R} . In many examples the representatives of i and i^\dagger will have an interpretation as creation and annihilation operators. In such cases we sometimes adhere to the curious, but universally accepted convention to make $f \mapsto a^\dagger(f)$ a linear operator, and $f \mapsto a(f)$ antilinear. In other words, we denote the representation by $a^\dagger : \mathcal{W}(T) \rightarrow \mathcal{R}$, and use the notational conventions

$$\begin{aligned} a_i^\dagger &= a^\dagger(i) \quad , \quad a_i = a^\dagger(i^\dagger) \\ a^\dagger(f) &= \sum_i f_i a_i^\dagger \quad , \quad a(f) = a^\dagger(f^\dagger) = \sum_i \overline{f_i} a_i \quad . \end{aligned} \quad (1.1.6)$$

The following Definition lists the properties of representations in which our main interest lies.

Definition 1.1.3. *A Hermitian representation of a Hermitian Wick algebra $\mathcal{W}(T)$ is a representation $\pi : \mathcal{W}(T) \rightarrow \mathcal{L}(\mathcal{R})$ by linear operators on a complex vector space with Hermitian form $\langle \cdot, \cdot \rangle : \mathcal{R} \times \mathcal{R} \rightarrow \mathbb{C}$ such that for all $\xi, \eta \in \mathcal{R}$, and $i \in I$:*

$$\langle \xi, \pi(i)\eta \rangle = \langle \pi(i^\dagger)\xi, \eta \rangle \quad . \quad (1.1.7)$$

*A representation π is called **positive**, if the form on the representation space \mathcal{R} is positive definite. It is called **bounded**, if it is positive, and $\pi(i)$ is bounded for all $i \in I$. It is called **collectively bounded** with bound β , if it is positive, and for all $\xi \in \mathcal{R}$:*

$$\sum_i \|\pi(i^\dagger)\xi\|^2 \leq \beta \langle \xi, \xi \rangle \quad . \quad (1.1.8)$$

Note that even when the hermitian form of \mathcal{R} is positive definite we have not assumed \mathcal{R} to be complete, i.e. a Hilbert space. If $\tilde{\mathcal{R}}$ denotes the completion of $(\mathcal{R}, \langle \cdot, \cdot \rangle)$, we

may consider $\pi(X)$ for $X \in \mathcal{W}(T)$ as an operator from the domain $\mathcal{R} \subset \tilde{\mathcal{R}}$ into $\tilde{\mathcal{R}}$. Each $\pi(X)$ will be closable with adjoint $\pi(X)^*$ relative to the scalar product of $\tilde{\mathcal{R}}$, and we have the operator inclusion $\pi(X^\dagger) \subset \pi(X)^*$. Note that \mathcal{R} is a common, dense, invariant domain for all the $\pi(X)$. Given a hermitian linear functional $\omega : \mathcal{W}(T) \rightarrow \mathbb{C}$, i.e. a functional with $\omega(X^\dagger) = \overline{\omega(X)}$, we get a hermitian form on $\mathcal{W}(T)$ by $\langle X, Y \rangle_\omega = \omega(X^\dagger Y)$. The representation acting on $\mathcal{W}(T)$ by left multiplication becomes hermitian with respect to this form. We call it the *cyclic*, or GNS- *representation* π_ω associated with ω . Obviously, this representation is positive if and only if ω is, i.e. iff $\omega(X^\dagger X) \geq 0$. Conversely, every vector in a positive representation defines a positive functional. Non-zero positive functionals need not exist on a Wick algebra, since the cone of elements $\sum X_i^\dagger X_i$ may contain all elements of the form “ $-Y^\dagger Y$ ”. Other related such “singular features” are at the heart of the representation theoretic problems for non-compact quantum groups, see e.g. [Wo4] and [vDa].

When a representation π is bounded, we may study the C^* -algebra generated by $\pi(\mathcal{W}(T))$ in $\mathcal{B}(\mathcal{R})$. It is clear that representations of this algebra in turn produce bounded representations of $\mathcal{W}(T)$. Therefore we may ask whether every bounded representation of $\mathcal{W}(T)$ arises in this way from a single representation. The following definition describes this situation.

Definition 1.1.4. *The universal bounded representation of a Wick algebra $\mathcal{W}(T)$ is a C^* -algebra, denoted $\mathbf{W}(T)$, which is generated by elements $i \in I$ satisfying the relations (1.1.1), with the following universal property: If $\pi : \mathcal{W}(T) \rightarrow \mathcal{B}(\mathcal{R})$ is a bounded representation on a Hilbert space \mathcal{R} , there is a unique C^* -algebra representation $\hat{\pi} : \mathbf{W}(T) \rightarrow \mathcal{B}(\mathcal{R})$ such that $\pi(i) = \hat{\pi}(i)$. The universal collectively bounded representation is defined analogously.*

The definition fits the pattern of a C^* -algebra presented by generators and relations [Bla,KW,JSW2,SW]. The existence of a universal bounded representation is equivalent to the following two conditions: firstly, there must be *some* bounded representation, and secondly, for each generator $i \in I$, there must be a number $\beta_i < \infty$ such that the uniform bound $\|\pi(i)\| \leq \beta_i$ holds in *every* bounded representation. If only the first condition holds, one can still find a universal representation in the class of “inverse limits” of C^* -algebras [Phi], or LMC*-algebras [Smü]. Since in all the examples we study below either both conditions hold or none of them, we will not elaborate on this structure.

1.2. Examples.

In this section we describe the principal examples, and classes of examples from the literature which motivated our study. We will repeatedly refer to these examples in the main parts of the paper, supplementing them by further examples illustrating more technical points. The descriptions given in this section are accompanied by brief indications of the new results we have obtained in each case, and where to find them in this paper.

The coherent representations λ_φ , which will be defined explicitly in the next section, are defined by $\lambda_\varphi(i^\dagger)\Omega = \varphi_i\Omega$, where $\varphi_i \in \mathbb{C}$, and Ω is a cyclic vector. For $\varphi_i \equiv 0$ we get the Fock representation λ_0 .

Example 1.2.1: q -canonical commutation relations.

We fix a real constant q , and set

$$i^\dagger j = \delta_{ij} \mathbb{I} + q j i^\dagger, \quad (1.2.1)$$

where the index set $I \ni i, j$ is arbitrary. Usually these relations are written in terms of “creation” operators a_j^\dagger and “annihilation operators” a_i as

$$a_i a_j^\dagger = \delta_{ij} \mathbb{I} + q a_j^\dagger a_i, \quad (1.2.1')$$

For $q = 1$ we have the canonical commutation relations for Bosons, for $q = -1$, we have the canonical anticommutation relations for Fermions. The relations (1.2.1') have been suggested by Greenberg [Gre] as an interpolation between Bose and Fermi statistics. Independently they were introduced by Bożejko and Speicher [BS1], who showed Fock positivity in this case (see also [Zag,Fiv,DN]). In the range $|q| < \sqrt{2} - 1$ the universal bounded representation was determined in [JSW1]. Coherent states were studied in [JW], generalizing earlier work [BEvGJ] on the case $q = 0$.

Example 1.2.2: Temperley-Lieb-Wick relations.

Superficially these relations look like the q -canonical commutation relations, but differ in the positioning of the indices in the term containing q :

$$i^\dagger j = \delta_{ij} \mathbb{I} + q i j^\dagger, \quad (1.2.2)$$

for $i, j = 1, \dots, d$. These relations were first studied in [JSW1]. We constructed the universal bounded representation for $|q|d < (\sqrt{5} - 1)/2$, and showed the independence of the universal C^* -algebra of q . Apart from the range $q \geq -1/d$ positive representations can only exist for a discrete series characterized by $q^{2N} = (1 + qd)/(q(d + q))$, $N \in \mathbb{N}$. One of the remarkable features of these relations in comparison with Example 1.2.1 is the sensitive dependence of the representation theory on the number d of generators. The reason behind this will become completely clear in Section 2.2. The connection with the Temperley-Lieb algebra [TL,Jo1] will also be explained there, and we show positivity of the Fock representation for $q > -1/(2d)$.

Example 1.2.3: Twisted canonical (anti-)commutation relations.

These relations have been introduced by Pusz and Woronowicz [PW,Pus]. The Bosonic version [PW] depends on a parameter $0 < \mu < 1$, and is given by

$$i^\dagger j = \begin{cases} \mu j i^\dagger & \text{for } i \neq j \\ \mathbb{I} + \mu^2 j i^\dagger - (1 - \mu^2) \sum_{k < i} k k^\dagger & \text{for } i = j. \end{cases} \quad (1.2.3)$$

We caution the reader that we have changed the ordering of the indices relative to [PW]. This will facilitate the discussion of the relations with infinitely many generators. A typical feature of these relations is that, in the Fock representation $\widehat{\lambda}_0$, one has $\widehat{\lambda}_0(ij - \mu ji) = 0$ for $i > j$. In the work [PW] these relations are additional postulates. Here, in Theorem 3.2.1, we show that they are true in every bounded representation, and moreover, that all bounded irreducible representations are coherent.

The Fermionic version of twisted commutation relations in this sense was discussed by [Pus], and is given by

$$i^\dagger j = \begin{cases} -\mu ji^\dagger & \text{for } i \neq j \\ \mathbb{1} - ji^\dagger - (1 - \mu^2) \sum_{k < i} kk^\dagger & \text{for } i = j. \end{cases} \quad (1.2.4)$$

Here the additional relations which turn out to be satisfied in the Fock representation are $\widehat{\lambda}_0(i)^2 = 0$, and $\widehat{\lambda}_0(ij + \mu ji) = 0$ for $i > j$. In Section 3.2 we investigate to what extent these relations are also consequences of positivity, like their Bosonic counterparts. We find that there are representations not satisfying these relations, and parametrize all possibilities by showing that all irreducible representations must be coherent (see Theorem 3.2.3).

Example 1.2.4: The quantum group $S_\nu U(2)$.

Some of the quantum groups of Woronowicz also can be considered as Hermitian Wick algebras with additional relations. For example, $S_\nu U(2)$ is given by the relations [Wo1]

$$\left. \begin{aligned} \alpha\alpha^* &= \mathbb{1} - \nu^2\gamma^*\gamma \\ \gamma\gamma^* &= \mathbb{1} - \alpha^*\alpha \\ \alpha\gamma^* &= \nu\gamma^*\alpha \end{aligned} \right\} \quad (1.2.5)$$

$$\left. \begin{aligned} \gamma\gamma^* &= \gamma^*\gamma \\ \alpha\gamma &= \nu\gamma\alpha \end{aligned} \right\} \quad (1.2.6)$$

The separation into (1.2.5) and (1.2.6) is not made in [Wo1]; it is suggested in our context since only the relations (1.2.5) fit the pattern of Wick algebra commutation rules (the generators are α^* and γ^*). It turns out that this is a fruitful way of looking at $S_\nu U(2)$, since the two remaining relations are redundant. The following Proposition also shows how coherent representations may help to obtain quickly all irreducible representations of the C^* -algebra $S_\nu U(2)$.

Proposition 1.2.4. *Let α and γ be operators on a Hilbert space, defined on a dense domain invariant under α, α^*, γ , and γ^* , on which the relations (1.2.5) hold. Then α and γ are bounded, and if $\nu \neq 0$, relations (1.2.6) also hold. Moreover, all irreducible representations are coherent in the following sense: if $\{\alpha, \gamma, \alpha^*, \gamma^*\}$ is an irreducible set of bounded operators, the representation space contains a vector Ω such that $\alpha\Omega = \hat{\alpha}\Omega$, and $\gamma\Omega = \hat{\gamma}\Omega$, where $\hat{\alpha}, \hat{\gamma} \in \mathbb{C}$, and*

$$|\hat{\alpha}| = 1, \hat{\gamma} = 0 \quad , \quad \text{or} \quad \hat{\alpha} = 0, |\hat{\gamma}| = 1 \quad . \quad (1.2.7)$$

Proof. Note first that from either of the first two relations we get that α and γ are both bounded in every positive representation. Consider the operators

$$\begin{aligned} C &= \alpha\gamma - \nu\gamma\alpha \\ \text{and} \quad R &= \mathbb{I} - \alpha^*\alpha - \gamma^*\gamma \quad . \end{aligned} \tag{1.2.8}$$

Then we have to show that $C = R = 0$. A straightforward computation using the Wick relations (1.2.5) gives

$$C^*C = R(\mathbb{I} - R) \quad \text{and} \tag{1.2.9a}$$

$$CC^* = -\nu^2 R(\mathbb{I} + \nu^2 R) \quad . \tag{1.2.9b}$$

Since $R = R^*$ (1.2.9a) forces $0 \leq R \leq 1$. Hence, by (1.2.9b), $CC^* \leq 0$, and $C = 0$. Then, from (1.2.9a), R has spectrum $\{0, 1\}$, i.e. R is a projection. When $\nu \neq 0$, (1.2.9b) implies $R = 0$, thus proving the first statement. In the case $\nu = 0$ we can only conclude with (1.2.8) that α and γ are partial isometries with $\mathbb{I} = \alpha\alpha^* \geq \mathbb{I} - \alpha^*\alpha = \gamma\gamma^* \geq \gamma^*\gamma$.

It follows that the universal bounded representation of (1.2.5) is precisely the C*-algebra of $S_\nu U(2)$, as defined by Woronowicz. We compute all positive coherent representations of this Wick algebra: Suppose that $\omega(X\alpha) = \hat{\alpha}\omega(X)$, and $\omega(X\gamma) = \hat{\gamma}\omega(X)$ for some complex constants $\hat{\alpha}, \hat{\gamma}$. Then positivity of ω requires $\omega(C) = \omega(R) = 0$, and hence $\hat{\alpha}\hat{\gamma} = 0$, and $|\hat{\alpha}|^2 + |\hat{\gamma}|^2 = 1$. These conditions are also sufficient for ω to be positive: when $\hat{\gamma} = 0$, and $\hat{\alpha}$ has modulus 1, the representation is one dimensional with $\alpha = \hat{\alpha}\mathbb{I}$, and $\gamma = 0$. On the other hand, when $\hat{\alpha} = 0$, and $\hat{\gamma}$ has modulus 1, we first construct the Fock representation of the Wick algebra determined by $\alpha\alpha^* = (1 - \nu^2)\mathbb{I} + \nu^2\alpha^*\alpha$, with a vacuum vector Ω satisfying $\alpha\Omega = 0$. Then we explicitly define

$$\gamma \alpha^{*n} \Omega = \hat{\gamma} \nu^n \alpha^{*n} \Omega \quad .$$

One readily verifies that this is the coherent representation, and, comparing with the known irreducible representations of the C*-algebra of $S_\nu U(2)$ [Mei] we find the result. \square

Example 1.2.5: Braid relations.

Examples 1.2.1 and 1.2.3 have a common feature, namely the validity of the identity

$$\sum_{efh} T_{be}^{fc} T_{ak}^{he} T_{hl}^{mf} = \sum_{deh} T_{ad}^{eb} T_{eh}^{mc} T_{dk}^{lh} \quad , \quad \forall abc, klm \in I \quad , \tag{1.2.10}$$

where the index set I labelling the generators is finite. In Section 2.2 we will identify this as the familiar relation determining the braid group. This turns out to be a useful assumption on T_{ij}^{kl} in two very different contexts: it allows an improvement of the bounds insuring Fock positivity (see Section 2.6), and also allows the extension of the differential calculus of Section 4.1 from the tensor algebra $\mathcal{T}(\mathcal{H})$ to the involutive algebra $\mathcal{W}(T)$ (see Section 4.3). The simplest case are the relations

$$i^\dagger j = \delta_{ij} \mathbb{I} + q_{ij} j i^\dagger \quad , \tag{1.2.11}$$

with $q_{ij} \in \mathbb{C}$. When q_{ij} takes only values ± 1 , but the indices cannot be grouped into “bosonic” and “fermionic” (which would be equivalent to $q_{ij} = (1 + q_i + q_j - q_i q_j)/2$, $q_i = \pm 1$) one speaks of anomalous statistics. Speicher [Sp2] has used the known Fock positivity in this case to show it also for general real q_{ij} with $|q_{ij}| \leq 1$. Below in Theorem 2.6.2 we will extend this Fock positivity result to complex q_{ij} with $|q_{ij}| \leq 1$. An independent proof generalizing to all T satisfying equation (1.2.10) was given by Bożejko and Speicher [BS2] (see Section 2.6). We also remark that the braid condition (1.2.10) is assumed in the work of Baez [Bae].

Example 1.2.6: Clifford algebras.

Clifford algebras are defined by the relations $i^\dagger j + j i^\dagger = \delta_{ij} \mathbb{1}$, $i, j = 1, \dots, d \leq \infty$, and hence correspond to the limit $q \rightarrow -1$ in (1.2.1), or $\mu \rightarrow 1$ in (1.2.4). The Fock representation of this algebra is identical with the Fermi relations, i.e. the generators anticommute among themselves, not only with their adjoints. In a coherent representation one has $\lambda_\varphi(ij + ji) = 2\varphi_i \varphi_j \mathbb{1}$. In the universal bounded representation the elements $\theta_{ij} = ij + ji$ are central. Hence, in every irreducible representation, θ_{ij} is a multiple of the identity. It turns out [JW] that an irreducible representation with given matrix θ exists if and only if $\|\theta\| \leq 1$ as Hilbert space operator on \mathbb{C}^d . Moreover, for each θ there are at most two different irreducible representations, which are finite dimensional if $d < \infty$.

Example 1.2.7: A degenerate case.

We consider the relations

$$i^\dagger j = \delta_{ij} (\mathbb{1} - \sum_{k \in I} k k^\dagger) \quad , \quad (1.2.12)$$

where $I = \{1, \dots, d\}$ is a finite set. This relation has special properties with respect to several of the questions we will discuss below: it is the only choice of coefficients for which the Fock scalar product becomes completely degenerate for two or more “particles”; it allows the largest possible quadratic Wick ideal (see Section 3.1), and it is the only choice for which all products of coordinate differentials in the calculus of Section 4.1 are linearly independent. Moreover, it is in the two parameter family of commutation relations

$$i^\dagger j = \delta_{ij} \mathbb{1} + q j i^\dagger - \lambda \delta_{ij} \sum_{k \in I} k k^\dagger \quad , \quad (1.2.13)$$

which is characterized by the full symmetry with respect to unitary transformations of \mathcal{H} [Sha]. Here we will briefly consider the representation theory of (1.2.12), using direct methods.

Theorem 1.2.7. *Let π be an irreducible representation of the relations (1.2.12) in a Hilbert space \mathcal{R} , and let $R = \mathbb{1} - \sum_i i i^\dagger \in \mathcal{W}(T)$. Then there is a number c_π , $0 \leq c_\pi \leq 1/4$ such that*

$$\pi(R(\mathbb{1} - R)) = c_\pi \mathbb{1} \quad .$$

- (1) For $c_\pi = 0$, π has to be the Fock representation, for which $\dim \mathcal{R} = d + 1$

- (2) For $c_\pi = 1/4$, we have $\pi(i) = 2^{-1/2} v_i$, where the $v_i, i = 1, \dots, d$ satisfy the Cuntz relations [Cu1]

$$v_i^* v_j = \delta_{ij} \mathbb{I} \quad , \quad \text{and} \quad \sum_i v_i v_i^* = \mathbb{I} \quad .$$

- (3) For $0 < c_\pi < 1/4$, there is an irreducible representation $w_{ij}, i, j = 1, \dots, d$ of the Cuntz relations on d^2 generators in a Hilbert space \mathcal{R}_0 such that, up to a unitary equivalence, \mathcal{R} is the direct sum of $(d+1)$ copies of \mathcal{R}_0 , and the generators are given by the following $\mathcal{B}(\mathcal{R}_0)$ -valued $(d+1) \times (d+1)$ -matrix, which we write in terms of the parameters α, β with $\alpha^2 = 1/2 + (1/4 - c_\pi)^{1/2}$, and $\beta^2 = 1/2 - (1/4 - c_\pi)^{1/2}$:

$$\pi(i) = \begin{pmatrix} 0 & 0 & \dots \beta \dots & 0 \\ \alpha w_{i1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha w_{id} & 0 & \dots & 0 \end{pmatrix} \quad , \quad (1.2.14)$$

where β in the first row is in the $(i+1)^{\text{th}}$ position.

Moreover, the coherent representation λ_φ with cyclic vector Ω satisfying $\lambda_\varphi(i^\dagger)\Omega = \varphi_i\Omega$ is positive if and only if

$$\left(\sum_i |\varphi_i|^2 \right)^2 = c_\pi \leq \frac{1}{4} \quad .$$

Proof. One readily checks that $iR + Ri = i$, and, by taking adjoints $i^\dagger R + Ri^\dagger = i^\dagger$. Hence by induction on the degree of polynomials we get $XR = (\mathbb{I} - R)X$ for all odd polynomials in the generators i, i^\dagger , and $RY = YR$ for all even ones. It follows that $R(\mathbb{I} - R)$ is in the center of $\mathcal{W}(T)$. Hence $\pi(R(\mathbb{I} - R))$ is a multiple of the identity in every irreducible representation π .

(1) By Wick ordering we get $\pi(R(\mathbb{I} - R)) = \sum_{ij} \pi(ij)\pi(ij)^*$. Hence $c_\pi = 0$ implies $\pi(ij) = 0$ for all i, j . Hence only the Wick ordered monomials of the forms $\mathbb{I}, i, i^\dagger, ij^\dagger$ can be non-zero. Hence $\dim \pi(\mathcal{W}(T)) \leq (d+1)^2$. Moreover,

$$\sum_i \pi(i^\dagger R)^* \pi(i^\dagger R) = \pi(R(\mathbb{I} - R)R) = 0 \quad .$$

Hence $\pi(i^\dagger R) = 0$, and $\pi(R)$ is the projection onto the one-dimensional subspace of Fock vectors.

(2) For $c_\pi = 1/4$, we have $\pi(R) = (1/2)\mathbb{I}$, and the relations (1.2.12) become the Cuntz relations up to a trivial change in normalization.

(3) $\pi(R)$ has the eigenvalues α^2 and β^2 . Let P denote the eigenprojection for eigenvalue α^2 , and $\mathcal{R}_0 = P\mathcal{R}$. Then, since $iR = (\mathbb{I} - R)i$, $\pi(i)$ swaps the eigenspaces of R , i.e. $\pi(i) =$

$u_i + v_i$, where $u_i = (\mathbb{I} - P)\pi(i)P : \mathcal{R}_0 \rightarrow \mathcal{R}_0^\perp$, and $v_i = P\pi(i)(\mathbb{I} - P) : \mathcal{R}_0^\perp \rightarrow \mathcal{R}_0$. In terms of these operators the relations (1.2.12) become $u_i^* u_j = \alpha^2 \delta_{ij} P$, and $v_i^* v_j = \beta^2 \delta_{ij} (\mathbb{I} - P)$, and the definition of R becomes $\sum_i u_i u_i^* = \alpha^2 (\mathbb{I} - P)$, and $\sum_i v_i v_i^* = \beta^2 P$. Using the partial isometries $\alpha^{-1} u_j$, we can identify $\mathcal{R}_0^\perp = (\mathbb{I} - P)\mathcal{R}$ with the direct sum of d copies of \mathcal{R}_0 .

$$\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_0^\perp = \mathcal{R}_0 \oplus \bigoplus_{i=1}^d \alpha^{-1} u_j \mathcal{R}_0 \cong \bigoplus_{i=0}^d \mathcal{R}_0 \quad .$$

In this structure we have encoded the action of u_i . The action of v_i is given by the isometries

$$w_{ij} = (\beta^{-1} v_i)(\alpha^{-1} u_j) = c_\pi^{-1/2} P \pi(ij) P \quad .$$

One readily verifies that these now satisfy the Cuntz relations, and that $\pi(i) = u_i + v_i$ is given by the matrix shown in the Theorem. To see the irreducibility of w_{ij} , suppose that $X \in \mathcal{B}(\mathcal{R}_0)$ commutes with the w_{ij} and their adjoints. Then $\hat{X} = PXP + \alpha^{-2} \sum_i u_i X u_i^*$ commutes with all $\pi(i), \pi(i)^*$. Hence \hat{X} is a multiple of \mathbb{I} , and so is X .

The \mathcal{R}_0 -component of the cyclic vector Ω has to be coherent [BEvGJ,JW] for the Cuntz algebra generators w_{ij} . The other components have to be multiples of the \mathcal{R}_0 -component. It is known for which eigenvalues such coherent states exist [BEvGJ], and this is translated into the statement given in the Theorem. \square

Example 1.2.8: Infinitely many generators.

Definition 1.1.1 allows the set I labelling the generators to be infinite with the proviso that the sums in the basic relations (1.1.1) are always finite. This assumption was sufficient to make the definitions of the previous section, as well as the definition of Fock and coherent representations meaningful. We saw that a Wick algebra can also be thought of as being generated by the vector space \mathcal{H} , of which I is a linear basis. However, in some of the applications it is desirable to take \mathcal{H} as a (possibly infinite dimensional) Hilbert space, with the set I as an orthonormal basis. In that case we will take the sums in (1.1.3) to be convergent in the norm of \mathcal{H} . All tensor products are to be read as Hilbert space tensor products, which makes the tensor algebra $\mathcal{T}(\mathcal{H}^\dagger, \mathcal{H})$ an involutive Banach algebra. We next have to make sense of the operator \tilde{T} in (1.1.5). The assumption we have made on its matrix elements $T_{ij}^{k\ell}$ are *not* sufficient to guarantee that \tilde{T} is bounded. This is illustrated by Example 1.2.3, where $\|\tilde{T}\|$ grows like $(1 - \mu^2)\sqrt{d}$ for the relations with $d < \infty$ generators. However, if we assume for the moment that $\|\tilde{T}\| < \infty$, the operator $f^\dagger \otimes g \mapsto \langle f, g \rangle \mathbb{I} + \tilde{T}(f^\dagger \otimes g)$ is also bounded, and its graph generates an ideal \mathcal{J} , such that $\mathcal{W}(T) := \mathcal{T}(\mathcal{H}^\dagger, \mathcal{H})/\mathcal{J}$ is a Banach algebra with involution. Of course, as for finitely many generators, this algebra may fail to have non-zero positive linear functionals.

Example 1.2.9: Relations with small coefficients T .

In this case the relations can be considered as a small perturbation [JSW1] of the relations with $T = 0$. The algebra $\mathcal{W}(0)$ has a universal bounded representation $\mathbf{W}(0)$, known as the Cuntz-Toeplitz algebra. It has a faithful representation as the algebra of creation

and annihilation operators in the full Fock space over $\mathcal{H} = \ell^2(I)$, i.e. in the direct sum of the n -fold (unsymmetrized) tensor products of \mathcal{H} with itself. We showed in [JSW1] that for sufficiently small T , $\mathbf{W}(T)$ exists, and is canonically isomorphic to $\mathbf{W}(0)$. From the form of this isomorphism we also get Fock positivity, and the faithfulness of the Fock representation. The precise meaning of “small T ” in this context will be given below in Section 2.4.

Example 1.2.10: Counterexamples.

Here we collect a few elementary examples to demonstrate some of the phenomena that can occur in Wick algebra theory.

1. A Wick algebra without any positive representations

Take two generators i, j with $i^\dagger i = j^\dagger j = \mathbb{I}$, and $i^\dagger j = \mu i i^\dagger$. The first two relations make i, j non-zero co-isometries in every positive representation, i.e. $\|i\| = \|j\| = 1$. Then the last equation gives a contradiction as soon as $|\mu| > 1$. One can also see this in a more algebraic way by observing that

$$i^\dagger(i^\dagger - j^\dagger)(i - j)i + (\mu - 1)\mathbb{I} = 0 \quad .$$

Hence, for $\mu > 1$, we find that \mathbb{I} is both positive and negative, and consequently vanishes in every positive representation.

2. A Wick algebra with positive, but without bounded representations

Take the canonical commutation relations (Example 1.2.1 with $q = 1$).

3. A Wick algebra with a universal bounded representation, whose Fock representation is not positive.

Take Woronowicz’ $S_\nu U(2)$ as described in Example 1.2.4. The universal bounded representation is Woronowicz’ C^* -algebra $S_\nu U(2)$. On the other hand, the element $C = \alpha\gamma - \nu\gamma\alpha$ gives a negative Fock expectation $\omega_0(CC^*) = -\nu^2(1 + \nu^2)$.

4. A Wick algebra for which the family of positive coherent representations is not faithful.

Take the Clifford algebras of Example 1.2.5. With $\theta_{ij} = ij + ji$, we have $\lambda_\varphi(\theta_{ij}\theta_{k\ell} - \theta_{ik}\theta_{j\ell}) = 0$ in any coherent representation λ_φ . But since there are irreducible representations in which θ is an arbitrary (but small) scalar symmetric matrix [JW], this identity does not hold in $\mathcal{W}(T)$.

1.3. Coherent representations.

There is an important set of representations of a Wick algebra $\mathcal{W}(T)$ which is intimately connected with the Wick ordering process: Let $\mathcal{T}(\mathcal{H}) = \mathbb{C}\mathbb{I} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \cdots$ denote the tensor algebra over \mathcal{H} . Let $\varphi : \mathcal{H} \rightarrow \mathbb{C}$ denote a conjugate linear functional on \mathcal{H} . For example, we could have $\varphi \in \mathcal{H}$, and consider the functional $f \mapsto \langle f, \varphi \rangle$. We will use this

notation even if there is no $\varphi \in \mathcal{H}$ allowing us to read $\langle f, \varphi \rangle$ as a bona fide inner product. Then we define a representation $\lambda_\varphi : \mathcal{W}(T) \rightarrow \mathcal{L}(\mathcal{T}(\mathcal{H}))$ by

$$\begin{aligned}\lambda_\varphi(f)X &= f \otimes X \\ \lambda_\varphi(f^\dagger)\mathbb{I} &= \langle f, \varphi \rangle \mathbb{I}\end{aligned}\tag{1.3.1}$$

for all $f \in \mathcal{H}$. Thus $\lambda_\varphi(f)$ acts on the tensor algebra $\mathcal{T}(\mathcal{H})$ by left multiplication, and according to our notational conventions we could also write $\lambda_\varphi(f)X = f \otimes X$. The crucial point is that we do not need to define $\lambda_\varphi(f^\dagger)X$ for $X \neq \mathbb{I}$. To see this, consider the following rule, which is automatic from our axiomatic setup:

$$\begin{aligned}\lambda_\varphi(f^\dagger)g_1 \otimes \cdots \otimes g_n &= \lambda_\varphi(f^\dagger)\lambda_\varphi(g_1)g_2 \otimes \cdots \otimes g_n \\ &= \langle g_1, f \rangle g_2 \otimes \cdots \otimes g_n \\ &\quad + \sum_{ijkl} T_{ij}^{kl} \lambda_\varphi(\ell) \lambda_\varphi(k^\dagger) g_2 \otimes \cdots \otimes g_n \quad .\end{aligned}\tag{1.3.2}$$

On the right hand side $\lambda_\varphi(k^\dagger)$ is applied to a tensor product of only $n - 1$ factors, so this is an inductive formula allowing us to extend λ_φ to tensors in $\mathcal{T}(\mathcal{H})$ of arbitrary length n . Note that we have not yet imposed an hermitian inner product on $\mathcal{T}(\mathcal{H})$. There is, however, only one way of introducing it:

Proposition 1.3.1. *Let $\mathcal{W}(T)$ be a Hermitian Wick algebra, and let $\varphi : \mathcal{H} \rightarrow \mathbb{C}$ be a conjugate linear functional. Then*

- (1) *There is a unique hermitian (but not necessarily positive semidefinite) inner product $\langle \cdot, \cdot \rangle_{T, \varphi}$ on $\mathcal{T}(\mathcal{H})$ with $\langle \mathbb{I}, \mathbb{I} \rangle_{T, \varphi} = 1$, making λ_φ a hermitian representation.*
- (2) *There is a unique hermitian linear functional ω_φ on $\mathcal{W}(T)$ such that $\omega_\varphi(\mathbb{I}) = 1$, and*

$$\omega_\varphi(fX) = \langle \varphi, f \rangle \omega_\varphi(X) \quad \text{for all } f \in \mathcal{H}, \text{ and } X \in \mathcal{W}(T).$$

(3)

$$\begin{aligned}\langle F, G \rangle_{T, \varphi} &= \omega_\varphi(F^\dagger G) \quad \text{for all } F, G \in \mathcal{T}(\mathcal{H}), \text{ and} \\ \omega_\varphi(X) &= \langle \mathbb{I}, \lambda_\varphi(X) \mathbb{I} \rangle_{T, \varphi} \quad \text{for all } X \in \mathcal{W}(T).\end{aligned}$$

- (4) *The inner product $\langle \cdot, \cdot \rangle_{T, \varphi}$ is positive semidefinite if and only if $\omega_\varphi(X^\dagger X) \geq 0$ for all $X \in \mathcal{W}(T)$.*

Proof. We begin by showing (2). Because ω_φ is hermitian we must also have $\omega_\varphi(Xg^\dagger) = \langle g, \varphi \rangle \omega_\varphi(X)$. The existence and uniqueness of ω_φ follows from the uniqueness of the Wick ordering process. The Wick ordered monomials $i_1 \cdots i_n j_1^\dagger \cdots j_m^\dagger$ form a basis of $\mathcal{W}(T)$, on which ω_φ is explicitly defined as

$$\omega_\varphi(i_1 \cdots i_n j_1^\dagger \cdots j_m^\dagger) = \prod_{k=1}^n \langle \varphi, i_k \rangle \prod_{\ell=1}^m \langle j_\ell, \varphi \rangle \quad .$$

Suppose $\langle \cdot, \cdot \rangle$ is an inner product with the properties specified in (1). Then

$$\begin{aligned} \langle \mathbb{1}, \lambda_\varphi(i_1 \cdots i_n j_1^\dagger \cdots j_m^\dagger) \mathbb{1} \rangle &= \langle \lambda_\varphi(i_n^\dagger \cdots i_1^\dagger) \mathbb{1}, \lambda_\varphi(j_1^\dagger \cdots j_m^\dagger) \mathbb{1} \rangle \\ &= \prod_{k=1}^n \langle \varphi, i_k \rangle \prod_{\ell=1}^m \langle j_\ell, \varphi \rangle \quad . \end{aligned}$$

by inductive application of the formula $\lambda_\varphi(f^\dagger) \mathbb{1} = \langle f, \varphi \rangle \mathbb{1}$. Since Wick ordered monomials form a basis, this proves the second part of (3). Hence

$$\begin{aligned} \langle i_1 \cdots i_n, j_1 \cdots j_m \rangle &= \langle \mathbb{1}, \lambda_\varphi(i_n^\dagger \cdots i_1^\dagger j_1 \cdots j_m) \mathbb{1} \rangle \\ &= \omega_\varphi((i_1 \cdots i_n)^\dagger (j_1 \cdots j_m)) \quad , \end{aligned}$$

which proves the first formula in (3), and hence (1), the existence and uniqueness of $\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_{T, \varphi}$. (4) is immediate from the first equation in (3). \square

It is interesting to note that in all these representations $\lambda_\varphi(i)$ is the same operator on $\mathcal{T}(\mathcal{H})$. Therefore, we may recover T and φ from the inner product $\langle \cdot, \cdot \rangle_{T, \varphi}$.

We will call λ_φ the *coherent representation*, and ω_φ the *coherent state* associated with φ . The coherent state and representation with $\varphi = 0$ are called the *Fock state* and the *Fock representation* of $\mathcal{W}(T)$. We use our notation $\langle \cdot, \cdot \rangle_{T, \varphi}$ in such a way that zeros in the index are suppressed. Thus $\langle \cdot, \cdot \rangle_T \equiv \langle \cdot, \cdot \rangle_{T, 0}$ is the Fock inner product, and $\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_{0, 0}$ is the Fock scalar product for the relations with $T = 0$, i.e. $i^\dagger j = \delta_{ij}$. A moment's reflection shows that this is the same as the inner product of $\mathcal{T}(\mathcal{H})$ inherited by the canonical inner product on \mathcal{H} . The completion of $\mathcal{T}(\mathcal{H})$ in the associated norm is the so-called “full Fock space”.

In general the inner product $\langle \cdot, \cdot \rangle_{T, \varphi}$ may be degenerate. For example, for the canonical commutation relations and the Fock state ω_0 , $f_1 f_2 - f_2 f_1$ has zero norm. Dividing out such vectors we obtain a new, usually simpler representation. It coincides with the cyclic (GNS-) representation associated with the coherent state ω_φ .

Definition 1.3.2. *Let $\varphi : \mathcal{H} \rightarrow \mathbb{C}$ be a conjugate linear functional, and let $\mathcal{T}(\mathcal{H})/(T, \varphi)$ denote the quotient of $\mathcal{T}(\mathcal{H})$ by the null space of the inner product $\langle \cdot, \cdot \rangle_{T, \varphi}$. Then the representation λ_φ of $\mathcal{W}(T)$ lifts to a representation on $\mathcal{T}(\mathcal{H})/(T, \varphi)$, and if λ_φ is bounded, it also extends to the completion of $\mathcal{T}(\mathcal{H})/(T, \varphi)$ with respect to $\langle \cdot, \cdot \rangle_{T, \varphi}$. In either case this representation will be called the **separated coherent representation** associated with φ , and will be denoted by $\widehat{\lambda}_\varphi$.*

The following simple observation gives one reason why coherent representations are useful.

Proposition 1.3.3. *Suppose that the coherent representation λ_φ of $\mathcal{W}(T)$ is bounded. Then $\widehat{\lambda}_\varphi$ is irreducible.*

Proof. When the coherent representation is positive, the coherent functional ω_φ is uniquely characterized by

$$\omega_\varphi((f - \langle \varphi, f \rangle)(f - \langle \varphi, f \rangle)^\dagger) = 0 \quad .$$

This property would also hold for any component in a convex decomposition of ω_φ into positive functionals. Hence ω_φ is pure, and consequently its GNS-representation $\widehat{\lambda}_\varphi$ is irreducible by a standard result [BR]. \square

1.4. A characterization of the Fock representation.

The Fock representation plays a distinguished rôle, as shown by the following Theorem. It generalizes the Wold decomposition in single operator theory [NF], which provides a canonical direct sum decomposition for an arbitrarily given isometry as a sum of a unilateral shift (with multiplicity) and a unitary. An isometry is, after all, the very simplest case of a relation of the form (1.1.1): we take a single generator, and all coefficients $T_{ij}^{k\ell} = 0$, so that $i^\dagger i = \mathbb{I}$. The Fock representation of this relation is indeed the unilateral shift. The classical Wold decomposition now allows us to decompose an arbitrary representation π as $\pi = \pi_0 \oplus \pi_1$, where π_0 is a multiple of the Fock representation, and the “unitary” subrepresentation π_1 is characterized by the property that every vector in the representation space can be given an arbitrarily long “iteration history” with respect to the operators $\pi_1(i)$. In the following Theorem this statement is extended verbatim to the general case of Wick algebra commutation relations. In the following it is understood that an empty product $\pi(i_1 \cdots i_n)$ with $n = 0$ is the identity.

Theorem (Wold decomposition) 1.4.1. *Let $\pi : \mathcal{W}(T) \rightarrow \mathcal{B}(\mathcal{R})$ be a bounded Hermitian representation of a Hermitian Wick algebra on a Hilbert space \mathcal{R} . Let*

$$\begin{aligned} \mathcal{N} &= \{ \varphi \in \mathcal{R} \mid \forall_i \pi(i^\dagger) \varphi = 0 \} \\ \mathcal{R}_0 &= \overline{\text{lin}} \{ \pi(i_1 \cdots i_n) \varphi \mid n \geq 0, i_1, \dots, i_n \in I, \varphi \in \mathcal{N} \} \\ \mathcal{R}_1 &= \bigcap_{n \geq 0} \overline{\text{lin}} \{ \pi(i_1 \cdots i_n) \psi \mid i_1, \dots, i_n \in I, \psi \in \mathcal{R} \} \quad . \end{aligned}$$

Then $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1$, and \mathcal{R}_0 and \mathcal{R}_1 are invariant under $\pi(\mathcal{W}(T))$ (either subspace may be $\{0\}$). Moreover, the representation π restricted to \mathcal{R}_0 is a multiple of the separated Fock representation with multiplicity $\dim \mathcal{N}$.

Proof. Let $\xi, \eta \in \mathcal{N}$. Then

$$\langle \xi, \pi(X) \eta \rangle = \langle \xi, \eta \rangle \omega_0(X)$$

by definition of the Fock state. Let η_α , $\alpha = 1, \dots, \dim \mathcal{N}$ be an orthonormal basis in \mathcal{N} . Then the cyclic subspaces $\mathcal{R}_{0,\alpha} = \overline{\text{lin}} \pi(\mathcal{W}(T))\eta_\alpha$ are orthogonal, and restricted to each π is the Fock representation. The sum of these spaces is \mathcal{R}_0 .

Now let

$$\begin{aligned}\mathcal{R}_0^{(n)} &= \overline{\text{lin}} \{ \pi(j_1 \cdots j_m) \varphi \mid m < n, j_1, \dots, j_m \in I, \varphi \in \mathcal{N} \} \\ \mathcal{R}_1^{(n)} &= \overline{\text{lin}} \{ \pi(i_1 \cdots i_n) \psi \mid i_1, \dots, i_n \in I, \psi \in \mathcal{R} \} \quad .\end{aligned}$$

Then since for $m < n$ Wick ordering of $i_n^\dagger \cdots i_1^\dagger j_1 \cdots j_m$ leaves at least one factor i^\dagger to the right of every monomial, we have

$$\langle \pi(j_1 \cdots j_m) \varphi, \pi(i_1 \cdots i_n) \psi \rangle = 0$$

for $\varphi \in \mathcal{N}$. Hence $\mathcal{R}_0^{(n)} \subset (\mathcal{R}_1^{(n)})^\perp$ for all n .

We claim that even $\mathcal{R}_0^{(n)} = (\mathcal{R}_1^{(n)})^\perp$. For this we need to show that $\psi \in (\mathcal{R}_1^{(n)})^\perp$, or, equivalently,

$$\pi(i_n^\dagger \cdots i_1^\dagger) \psi = 0 \quad (*)$$

for all $i_1, \dots, i_n \in I$, implies $\psi \in \mathcal{R}_0^{(n)}$. Since the decomposition $\mathcal{R} = \mathcal{R}_0 \oplus (\mathcal{R}_0)^\perp = \left(\bigoplus_{\alpha=1}^{\dim \mathcal{N}} \mathcal{R}_{0,\alpha} \right) \oplus (\mathcal{R}_0)^\perp$ is invariant under π , it suffices to show this for ψ in each summand separately. Assume first that $\psi \in \mathcal{R}_0^\perp$ satisfies (*). Then for all i_1, \dots, i_{n-1} we have $\pi(i_{n-1}^\dagger \cdots i_1^\dagger) \psi \in \mathcal{N} \cap \mathcal{R}_0^\perp = \{0\}$. Hence (*) also holds for $n-1$. Proceeding by downwards induction we find $\psi = 0$. Assume next that $\psi \in \mathcal{R}_{0,\alpha}$ for one of the Fock summands, generated from a vector $\eta_\alpha \in \mathcal{N}$. Then for $m \geq n$ we have $\pi(i_1 \cdots i_n) \eta_\alpha \in \mathcal{R}_1^{(n)}$. By construction of the Fock representation these vectors span the orthogonal complement of $\mathcal{R}_0^{(n)}$ so we have $\psi \in \mathcal{R}_0^{(n)}$.

Since $\mathcal{R}_0 = \overline{\text{lin}} \bigcup_{n \in \mathbb{N}} \mathcal{R}_0^{(n)}$, and $\mathcal{R}_1 = \bigcap_{n \in \mathbb{N}} \mathcal{R}_1^{(n)}$, the complementarity of the spaces $\mathcal{R}_i^{(n)}$ for each n implies $\mathcal{R}_0 = \mathcal{R}_1^\perp$. \square

2. BOUNDEDNESS AND POSITIVITY

2.1. The coefficients $T_{ij}^{k\ell}$ as an operator.

One might expect that positivity and boundedness of the Fock representation of a Wick algebra depend on positivity and boundedness of \tilde{T} , considered as an operator between tensor product Hilbert spaces. However, this is *not* the case. To see this, let us compute the Fock inner product of the “two-particle” space $\mathcal{H} \otimes \mathcal{H}$, i.e.

$$\langle ij, k\ell \rangle_T := \omega_0(j^\dagger i^\dagger k\ell) = \delta_{ik} \delta_{jl} + T_{ik}^{\ell j} \quad . \quad (2.1.1)$$

We would like to consider this as a matrix element of an operator in the 2-fold Hilbert space tensor product $\mathcal{H} \otimes \mathcal{H}$ with its natural inner product $\langle \cdot, \cdot \rangle$. Thus we introduce an operator $T : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ with the matrix elements

$$\langle ij|T|k\ell\rangle = T_{ik}^{\ell j} \quad , \quad (2.1.2)$$

or, equivalently,

$$T k \otimes \ell = \sum_{ij} T_{ik}^{\ell j} i \otimes j \quad . \quad (2.1.2')$$

Then we can write

$$\langle ij, k\ell\rangle_T = \langle ij|\mathbb{I} + T|k\ell\rangle = \langle ij, P_2 k\ell\rangle \quad . \quad (2.1.3)$$

Hence the Fock inner product of order 2 is positive, if and only if $P_2 = \mathbb{I} + T \geq 0$. Notice in equation (2.1.3) that the indices $k\ell$ are associated with the domain of T , and ij with its range, whereas the pair ℓj corresponds to the domain of \tilde{T} . Thus in contrast to the transposition operation, which would just swap domain and range indices, only one index has changed sides, i.e. T is obtained from \tilde{T} by transposing only in one tensor factor. Since the transposition operation is not completely bounded or completely positive, the norm and positivity properties of T may be quite different from those of \tilde{T} . As the above computation (2.1.2) shows, it is T rather than \tilde{T} , which will be relevant for questions like Fock positivity.

The formula $P_2 = \mathbb{I} + T$ has an extension to higher tensor powers. We define the operators P_n on the n -fold tensor product space $\mathcal{T}_n(\mathcal{H}) = \mathcal{H}^{\otimes n}$ such that

$$\langle i_1 \cdots i_n, j_1 \cdots j_n \rangle_T =: \langle i_1 \cdots i_n, P_n j_1 \cdots j_n \rangle \quad . \quad (2.1.4)$$

Note that with respect to $\langle \cdot, \cdot \rangle_T$ the subspaces $\mathcal{T}_n(\mathcal{H})$ and $\mathcal{T}_m(\mathcal{H})$, for $n \neq m$, are orthogonal as for $\langle \cdot, \cdot \rangle$. Positivity of the Fock inner product means that $P_n \geq 0$ for all n . In order to state the formula for P_n we introduce the notation

$$X_i(f_1 \otimes f_2 \cdots f_i \otimes f_{i+1} \otimes \cdots f_n) = (f_1 \otimes f_2 \cdots \otimes X(f_i \otimes f_{i+1}) \otimes \cdots \otimes f_n) \quad (2.1.5)$$

where $i = 1, \dots, n-1$, and X is any operator on a product Hilbert space $\mathcal{H} \otimes \mathcal{H}$. Thus X_i becomes an operator on the n -fold tensor product $\mathcal{T}_n(\mathcal{H}) = \mathcal{H} \otimes \mathcal{H} \cdots \otimes \mathcal{H}$.

Lemma 2.1.1. *For any bounded operator T on $\mathcal{H} \otimes \mathcal{H}$, define a sequence $P_n(T)$ of bounded operators on $\mathcal{T}_n(\mathcal{H})$ by the inductive formula $P_1 = \mathbb{I}$,*

$$P_{n+1}(T) = (\mathbb{I} \otimes P_n(T))(\mathbb{I} + T_1 + T_1 T_2 + \cdots + T_1 \cdots T_n) \quad . \quad (2.1.6)$$

Then if T is given by equation (2.1.2), P_n from equation (2.1.4) is equal to $P_n(T)$.

Proof. P_n describes the difference between the Fock inner products $\langle \cdot, \cdot \rangle_T$ and $\langle \cdot, \cdot \rangle_0 \equiv \langle \cdot, \cdot \rangle$. Denoting by λ_0 and μ the Fock representations of the respective relations on $\mathcal{T}(\mathcal{H})$, we

have from (1.3.1): $\lambda_0(i)X = \mu(i)X = i \otimes X$, and $\mu(i^\dagger)\mu(j) = \delta_{ij}$. $\lambda_0(i^\dagger)$ is given by equation (1.3.2). Our first task will be to express it in terms of $\mu(i^\dagger)$, in the form

$$\lambda_0(i^\dagger)X = \mu(i^\dagger)R_n X \quad \text{for } X \in \mathcal{T}_n(\mathcal{H}) \quad , \quad (*)$$

where R_n is an operator on $\mathcal{T}_n(\mathcal{H})$ to be determined in terms of T by an inductive formula. With $X \in \mathcal{T}_n(\mathcal{H})$ we have

$$\begin{aligned} \mu(i^\dagger)R_{n+1}\mu(j)X &= \lambda_0(i^\dagger)\lambda_0(j)X \\ &= \delta_{ij}X + \sum_{k\ell} T_{ij}^{k\ell} \lambda_0(\ell) \lambda_0(k^\dagger) X \\ &= \delta_{ij}X + \sum_{k\ell} T_{ij}^{k\ell} \mu(\ell) \mu(k^\dagger)R_n X \\ &= \mu(i^\dagger)\mu(j)X + \sum_{k\ell, nm} T_{nm}^{k\ell} \mu(i^\dagger)\mu(n)\mu(\ell)\mu(k^\dagger)\mu(m^\dagger) \mu(j)R_n X \\ &= \mu(i^\dagger)\left(\mathbb{1} + T(\mathbb{1} \otimes R_n)\right)\mu(j)X \quad , \end{aligned}$$

where at the last step we used the identity $\mu(j)R_n = (\mathbb{1}_{\mathcal{H}} \otimes R_n)\mu(j)$ for operators on $\mathcal{T}_n(\mathcal{H})$, and introduced the operator T from (1.1.2) in the form

$$T = \sum_{k\ell, nm} T_{nm}^{k\ell} \mu(n)\mu(\ell)\mu(k^\dagger)\mu(m^\dagger) \quad .$$

Hence we have proved the formula $(*)$ with

$$R_{n+1} = \mathbb{1} + T(\mathbb{1}_{\mathcal{H}} \otimes R_n) \quad ,$$

or $R_n = (\mathbb{1} + T_1 + T_1 T_2 + \cdots + T_1 \cdots T_{n-1})$.

The claim of the Lemma thus reduces to the formula $P_{n+1} = (\mathbb{1}_{\mathcal{H}} \otimes P_n)R_{n+1}$. But

$$\begin{aligned} \langle \mu(i)X, P_{n+1}\mu(j)Y \rangle &= \langle X, \lambda_0(i^\dagger)\lambda_0(j)Y \rangle_T = \langle X, \mu(i^\dagger)R_{n+1}\mu(j)Y \rangle_T \\ &= \langle X, P_n\mu(i^\dagger)R_{n+1}\mu(j)Y \rangle = \langle X, \mu(i^\dagger)(\mathbb{1} \otimes P_n)R_{n+1}\mu(j)Y \rangle \\ &= \langle \mu(i)X, (\mathbb{1} \otimes P_n)R_{n+1}\mu(j)Y \rangle \quad , \end{aligned}$$

which completes the proof. \square

2.2. The operator T in the examples.

It is instructive to compute the operator T in the examples of Section 1.2. In each example, the properties of T as an operator on a Hilbert space illuminates special features of the example.

For **Example 1.2.1**, the q -canonical commutation relations, we get

$$T|ij\rangle = q|ji\rangle \quad , \quad (2.2.1)$$

i.e. q times the flip operator. Thus in the Bose case ($q = 1$), $P_n(T)$, from equation (2.1.4) is proportional to the symmetrization projection. In the Fermi case ($q = -1$) each term gets a factor equal to the sign of the permutation so that P_n is proportional to the anti-symmetrization projection.

For **Example 1.2.2**, the Temperley-Lieb-Wick relations, we get

$$T|ij\rangle = q\delta_{ij} \sum_k |kk\rangle \quad . \quad (2.2.2)$$

In order for this sum to converge, we must assume that the number d of generators is finite. Then T is qd times a one-dimensional projection, so $\|T\| = |q|d$. This readily accounts for the d -dependence of the representation theory, which we remarked in [JSW1], without being able to pinpoint the reason for this dependence in comparison with Example 1.2.1. One easily checks that T satisfies the Temperley-Lieb relations [TL,Jo1]

$$\begin{aligned} T_1 T_2 T_1 &= q^2 T_1 \\ T^2 &= qd T \quad . \end{aligned} \quad (2.2.3)$$

As a generator of the Temperley-Lieb algebra it would be natural to consider $T' = (qd)^{-1}T$, which is a projection, and satisfies $T'_1 T'_2 T'_1 = d^{-2} T'_1$. This relation no longer contains the parameter q , from which we conclude that the analysis of the associated Temperley-Lieb algebra is not sufficient to decide the positivity questions we address in this paper. Note that while in Example 1.2.1 T always has both positive and negative eigenvalues, T in this example is positive for $q \geq 0$. Hence by Theorem 2.5.1, the Fock representation of these relations is positive for $q \geq 0$. Theorem 2.3.2 enlarges this range by $\|T\| = |q|d < 1/2$, so all together we have a positive Fock representation for $q > -1/(2d)$.

For **Example 1.2.3**, the twisted CCR and CAR, we have

$$T|ij\rangle = \begin{cases} \mu|ji\rangle & i < j \\ \mu^2|ii\rangle & i = j \\ -(1 - \mu^2)|ij\rangle + \mu|ji\rangle & i > j \end{cases} \quad (2.2.4)$$

in the Bosonic case (1.2.3), and

$$T|ij\rangle = \begin{cases} -\mu|ji\rangle & i < j \\ -|ii\rangle & i = j \\ -(1 - \mu^2)|ij\rangle - \mu|ji\rangle & i > j \end{cases} \quad (2.2.5)$$

in the Fermionic case (1.2.4). In both these cases T has the eigenvalues μ^2 and -1 , and $\|T\| = 1$.

For **Example 1.2.4**, i.e. $S_\nu U(2)$ with the relations (1.2.5), we get the eigenvalues 0 and $-(1 + \nu^2)$, so $\|T\| = 1 + \nu^2$. As noted above, and as is evident from (2.1.3) and the eigenvalue < -1 of T , Fock positivity fails in this example.

For **Example 1.2.5**, the braided case, we get the rather more transparent form $T_1 T_2 T_1 = T_2 T_1 T_2$ of equation (1.2.10), which justifies calling it a braid relation. For the special case of equation (1.2.11) we find

$$T|ij\rangle = q_{ij}|ji\rangle \quad . \quad (2.2.6)$$

Hence, in this case $\|T\| = \max_{ij} |q_{ij}|$.

Example 1.2.6, the case of Clifford algebras, is a special case of Example 1.2.1, so $T|ij\rangle = -|ji\rangle$.

The “degenerate” **Example 1.2.7** is simply $T = -\mathbf{I}$. This is equivalent to $P_2(T) = 0$, and $P_n(T) = 0$ for all $n \geq 2$ by Lemma 2.1.1. Note also that the braid relation is satisfied.

In **Example 1.2.8**, the case of infinitely many generators, we see that T may turn out to be unbounded: in Example 1.2.2 we have seen that $\|T\| = |q|d$ for the relations on d generators, hence for infinitely many generators T is not bounded. It cannot even be densely defined as an unbounded operator, as is evident from (2.2.2). On the other hand, $\|\tilde{T}\| = 1$ in this example. At the other extreme we have Example 1.2.3, where $\|\tilde{T}\|$ diverges, but T is a well-defined bounded operator. It is not clear whether boundedness of T , without the finiteness condition of Definition 1.1.1, would allow us to give a definition of the abstract Wick algebra $\mathcal{W}(T)$ on infinitely many generators. However, it suffices to define the Fock representation, assuming that it happens to be positive: from Lemma 2.1.1 we get the operators $P_n(T)$, which are non-negative definite by assumption. These define the inner product on $\mathcal{T}(\mathcal{H})$, and its completion, the Fock space. By construction, this space has a natural orthogonal decomposition into “ n -particle” spaces \mathcal{H}^n . As shown in the proof of Proposition 2.3.1 below, the operator $X \mapsto f \otimes X$ is bounded with respect to the Fock scalar products as an operator taking \mathcal{H}^n to \mathcal{H}^{n+1} (The bound is $\sum_{i=1}^n \|T\|^n$). Hence we get a well-defined closed operator $\hat{\lambda}_0(f)$ as the orthogonal sum of these bounded operators. The operators $\hat{\lambda}_0(f)$ and $\hat{\lambda}_0(g)^*$ have the space of vectors with finite particle number as a common dense and invariant domain. Clearly, the algebra generated by these operators coincides with $\hat{\lambda}_0(\mathcal{W}(T))$, whenever the latter can be defined with the methods of Section 1.

In **Example 1.2.9**, i.e. the case of small coefficients $T_{ij}^{k\ell}$, the operator T will also be small. The next section is devoted to showing that $\|T\|$ is, in fact, the appropriate measure of “smallness” for many questions.

2.3. Bounds for small T .

In this section we discuss conditions (in the general case) under which one can obtain Fock positivity or boundedness of the Fock representation. A crucial parameter in these problems is $\|T\|$, where T is given in terms of the coefficients $T_{ij}^{k\ell}$ by (2.1.2).

The following Proposition generalizes Lemma 4 in [BS1].

Proposition 2.3.1. *Let $\|T\| < 1$, and suppose that $P_n \geq 0$ for all n . Then in the Fock representation the operators $\lambda_0(i)$ are bounded with*

$$\|\lambda_0(i)\|^2 \leq \frac{1}{1 - \|T\|} \quad .$$

Proof. For bounded operators A, B, C with $A, B \geq 0$ and $A = BC$, we have $A^2 = AA^* = BCC^*B^* \leq \|C\|^2 B^2$, and, by the operator monotonicity of the square root [RS,Don], we have $A \leq \|C\| B$. Applying this to the inductive formula (2.1.6) with $A = P_{n+1}$, $B = \mathbb{1} \otimes P_n$, and using the norm estimate $\|C\| = \|\mathbb{1} + T_1 + T_1 T_2 + \cdots + T_1 \cdots T_{n-1}\| \leq (1 - \|T\|^n)/(1 - \|T\|) \leq (1 - \|T\|)^{-1}$, we get $P_{n+1} \leq (1 - \|T\|)^{-1} \mathbb{1} \otimes P_n$. Hence, for any vector $\Phi \in \mathcal{H}^{\otimes n}$ we have

$$\begin{aligned} \langle \Phi, \lambda_0(i)^* \lambda_0(i) \Phi \rangle_T &= \langle i \otimes \Phi, P_{n+1} (i \otimes \Phi) \rangle \\ &\leq (1 - \|T\|)^{-1} \langle i \otimes \Phi, (\mathbb{1} \otimes P_n) i \otimes \Phi \rangle \\ &= (1 - \|T\|)^{-1} \langle \Phi, \Phi \rangle_T \quad . \end{aligned}$$

□

We note that the bound on $\|T\|$ is best possible: for the Bose canonical commutation relations we have $\|T\| = 1$, and it is known (see e.g. [BR]) that these have no bounded representations. On the other hand, the criterion $\|T\| < 1$ is not a necessary condition for the boundedness of the a_i , as the example of the canonical anticommutation relations shows. In the next Theorem we use the inductive formula to get the positivity of P_n for sufficiently small T . Since the factors in the inductive formula do not commute, we cannot use it directly to show positivity of P_n . Instead, we show first that the P_n are non-singular, and then invoke analytic perturbation theory to get positivity. With a different argument for the non-singularity of P_n this idea has been used by [Zag,Fiv] to treat the q -canonical commutation relations, and by [BS2] in a much more general context.

Theorem 2.3.2. *For $\|T\| < 1/2$, $P_n \geq \varepsilon_n \mathbb{1} > 0$ for all n . For $\|T\| \leq 1/2$, $P_n \geq 0$ for all n .*

Proof. Assume $\|T\| < 1/2$. Then in the inductive formula

$$P_n = P_n(T) = (\mathbb{1} \otimes P_{n-1}(T)) \left(\mathbb{1} + \sum_{r=1}^{n-1} \prod_{i=1}^r T_i \right)$$

the sum in the second factor is bounded by $\sum_{r=1}^{n-1} \|T\|^r \leq \|T\| / (1 - \|T\|) < 1$. Hence

$$\|P_n(T)^{-1}\| \leq \frac{1 - \|T\|}{1 - \|2T\|} \|P_{n-1}(T)^{-1}\| \quad , \quad (2.3.1)$$

and $P_n(T)$ is invertible for all n .

Note that $P_n(\lambda T)$ is hermitian and invertible for all $\lambda \in [0, 1]$. The eigenprojection of $P_n(\lambda T)$ for the negative half axis can thus be computed by the same Cauchy integral in the analytic functional calculus for all λ , and is hence a norm continuous function of λ . Since this projection vanishes for $\lambda = 0$, it must be identically zero. Hence $P_n > 0$ for all n .

Since $P_n(T)$ is a polynomial in translates of T , it is clear that $\langle \varphi, P_n(\lambda T) \varphi \rangle$ is a continuous function of λ . If $\|T\| = 1/2$ this function is positive for $0 < \lambda < 1$, hence also for $\lambda = 1$. \square

We now show that, without further information about T , the above bound on $\|T\|$ is best possible.

Example 2.3.3. Consider the Wick algebra $\mathcal{W}(T)$ on two generators x, y with the relations

$$\begin{aligned} x^\dagger y &= y^\dagger x = 0 \\ x^\dagger x &= \mathbb{1} + \tau(xx^\dagger - yy^\dagger) \\ y^\dagger y &= \mathbb{1} - \tau(xx^\dagger - yy^\dagger) \quad , \end{aligned} \quad (2.3.2)$$

where $\tau \geq 0$ is a real parameter. Then $\|T\| = \tau$. For $\tau > 1/2$, $\mathcal{W}(T)$ has no bounded representations, and for $1/2 < \tau < 1$ the Fock representation is not positive.

Proof. The defining coefficients are

$$T_{ij}^{k\ell} = \tau(-1)^{j+k} \delta_{ij} \delta_{k\ell} \quad , \quad i, j, k, \ell = 1, 2 \quad .$$

Thus T is diagonal in the tensor product basis, and has eigenvalues $\pm\tau$. In particular, $\|T\| = \tau$. Consider the completely positive map

$$\Phi(a) = \frac{1}{2}(x^\dagger a x + y^\dagger a y) \quad .$$

By the relations it satisfies $\Phi(\mathbb{1}) = \mathbb{1}$, and for the special element $\Delta = (xx^\dagger - yy^\dagger)$ a short computation gives $\Phi(\Delta) = 2\tau\Delta$.

We show next that, for $\|T\| = \tau > 1/2$, the relations (2.3.2) have no bounded representations. If π were a bounded representation, $\pi(\Delta)$ would be bounded, and $2\tau \|\pi(\Delta)\| = \|\pi\Phi(\Delta)\| \leq \|\pi(\Delta)\|$. When $\tau > 1/2$, this implies $\pi(\Delta) = 0$. Thus $x := \pi(x^\dagger x) = \pi(yy^\dagger)$, and $x^2 = \pi(x x^\dagger y y^\dagger) = 0$. It follows that $x = 0$, hence $\pi(x) = \pi(y) = 0$ in contradiction with the Wick relations.

Now let $1/2 < \tau < 1$. Then if the Fock representation were positive, we could conclude by Proposition 2.3.1 that $\widehat{\lambda}_0(x), \widehat{\lambda}_0(y)$ are bounded, which is impossible by the preceding paragraph. \square

2.4. The universal bounded representation.

In this section we prove the criterion for the isomorphism between the universal bounded representation and the Cuntz-Toeplitz algebra for sufficiently small T . First we need a bound on the right hand side of the relations (1.1.1). We will think of it as an operator in block matrix form with respect to i, j . Thus we set

$$M_{ij}(a, b) = \sum_{k\ell} T_{ij}^{k\ell} a_\ell^* b_k, \quad (2.4.1)$$

where a_i, b_i are arbitrary collections of bounded operators (usually $b_i = a_i = \widehat{\lambda}_0(i^\dagger)$). The right hand side can also be written as a matrix

$$E_{ij}(a, b) = b_i a_j^*. \quad (2.4.2)$$

In this notation the relations (1.1.1) become $E(a, a) = \mathbb{1} + M(a, a)$, where $\mathbb{1}$ now refers to the identity in the d -fold direct sum of the space on which the a_i act. For any d -tuple of bounded operators a_i we define the norm

$$\|a\|^2 = \left\| \sum_k a_k^* a_k \right\|. \quad (2.4.3)$$

Proposition 2.4.1. *Suppose that $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ are tuples of operators with $\|a\|, \|b\| < \infty$. Let t_+, t_- denote the supremum and infimum of the spectrum of T , respectively. Then*

- (1) $\|E(a, b)\| \leq \|a\| \|b\|$,
- (2) $\|E(a, a)\| = \|a\|^2$,
- (3) $\|M(a, b)\| \leq \|T\| \|a\| \|b\|$, and
- (4) $t_- \|a\|^2 \mathbb{1} \leq M(a, a) \leq t_+ \|a\|^2 \mathbb{1}$.

Proof. (1) and (2) were proven in [JSW1], Lemma 10.

(3) Let \mathcal{R} denote the Hilbert space on which the a_i, b_i act. Let $\xi, \eta \in \mathcal{R}^I$. Then

$$\begin{aligned} \langle \xi, M(a, b)\eta \rangle &= \sum_{ij} \langle \xi_i, M_{ij}(a, b)\eta_j \rangle = \sum_{ijk\ell} \langle i\ell|T|jk \rangle \langle i\ell|R|jk \rangle \\ &= \text{tr}(TR) \quad , \end{aligned}$$

where R denotes the operator with matrix elements

$$\langle i\ell|R|jk \rangle = \langle a_\ell \xi_i, b_k \eta_j \rangle \quad .$$

Now by Lemma 2.4.2 below, R is trace class, with trace norm

$$\|R\|_1^2 \leq \left(\sum_{i\ell} \|a_\ell \xi_i\|^2 \right) \left(\sum_{kj} \|b_k \eta_j\|^2 \right) \leq \|\xi\|^2 \|\eta\|^2 \|a\|^2 \|b\|^2 \quad .$$

Hence

$$|\langle \xi, M(a, b)\eta \rangle| \leq \|T\| \|a\| \|b\| \|\xi\| \|\eta\| \quad .$$

(4) When $a = b$ and $\xi = \eta$, R is a positive trace class operator, hence $\langle \xi, M(a, a)\xi \rangle$ can be estimated above by the corresponding expression with $T = t_+ \mathbb{I}$, which is $t_+ \sum_{i\ell} \|a_\ell \xi_i\|^2 \leq t_+ \|a\|^2 \|\xi\|^2$. The lower bound follows analogously. \square

Lemma 2.4.2. *Let $d \leq \infty$, and let \mathcal{H} be a Hilbert space. For $i \in \mathbb{N}, 1 \leq i \leq d$, let $\psi_i, \varphi_i \in \mathcal{H}$. Suppose that $\|\psi\|^2 = \sum_i \|\psi_i\|^2 < \infty$, and $\|\varphi\|^2 = \sum_i \|\varphi_i\|^2 < \infty$. Then there is a unique trace-class operator R on \mathbb{C}^d with $\langle i|R|j \rangle = \langle \psi_i, \varphi_j \rangle$, which satisfies the estimate $\text{tr}|R| \leq \|\psi\| \|\varphi\|$.*

Proof. Consider the operator $V_\varphi : \mathbb{C}^d \rightarrow \mathcal{H}$ given by $V_\varphi|i\rangle = \varphi_i$. Then $V_\varphi V_\varphi^* = \sum_i |\varphi_i\rangle \langle \varphi_i|$. This positive operator has trace $\|\varphi\|^2$, hence V_φ is Hilbert-Schmidt class with Hilbert-Schmidt norm $\|V_\varphi\|_2^2 = \|\varphi\|^2$. With $R = V_\varphi^* V_\psi$, and a standard trace norm estimate we get $\text{tr}|R| = \|R\|_1 \leq \|V_\varphi\|_2 \|V_\psi\|_2 \leq \|\varphi\| \|\psi\|$. \square

The bounds in Proposition 1.1.3 imply the existence of universal representations in the following sense:

Proposition 2.4.3. *Let $T \leq t_+ \mathbb{I}$ with $t_+ < 1$, and suppose that there is a collectively bounded representation of $\mathcal{W}(T)$ in the sense of Definition 1.1.1. Then there is a universal collectively bounded representation with*

$$\beta = \left\| \sum_i \pi(ii^\dagger) \right\| \leq (1 - t_+)^{-1} \quad .$$

Proof. The tuple $a_i = \pi(i^\dagger)$, $i = 1, \dots, |I|$ satisfies the relations $E(a, a) = \mathbb{1} + M(a, a)$, and $\beta = \|a\|^2$. From Proposition 2.4.1(2,4) we get $\|a\|^2 \leq 1 + t_+ \|a\|^2$. \square

The following is the application of Theorem 9 in [JSW1] in the present context.

Theorem 2.4.4. *Let $d < \infty$, and let $t_\pm \in \mathbb{R}$ with $t_- \mathbb{1} \leq T \leq t_+ \mathbb{1}$. Suppose that*

$$\max\{|t_+|, |t_-|\}^2 < 1 - t_+ + t_- , \quad (2.4.4)$$

or, more specially, $\|T\| \leq \sqrt{2} - 1 \approx .414$, or $\|T\| \leq (\sqrt{5} - 1)/2 \approx .618$ with either $T \geq 0$ or $T \leq 0$. Then the universal bounded representation of $\mathcal{W}(T)$ exists, and $\mathbf{W}(T)$ is isomorphic to the Cuntz-Toeplitz algebra $\mathbf{W}(0)$.

Proof. We verify the premises of Theorem 9 in [JSW1]. From Proposition 2.4.3 it is clear that any solution to $E(a, a) = \mathbb{1} + M(a, a)$ must satisfy $\|a\|^2 \leq (1 - t_+)^{-1} \equiv \mu$. Moreover, this bound on an arbitrary tuple a implies that $\mathbb{1} + M(a, a) \leq \mu \mathbb{1}$. Under the same hypothesis $\mathbb{1} + M(a, a) \geq (1 + t_- \mu) \mathbb{1} \equiv \varepsilon \mathbb{1}$. Finally from Proposition 1.1.3 we get for $\|a\|^2, \|b\|^2 \leq \mu$ the Lipschitz bound $\|M(a, a) - M(b, b)\| \leq \|M(a - b, a)\| + \|M(b, a - b)\| \|T\| (\|a\| + \|b\|) \|a - b\| \leq \lambda \|a - b\|$ with $\lambda = 2\sqrt{\mu} \|T\|$. The hypothesis of Theorem 9 in [JSW1] now requires $\lambda < 2\sqrt{\varepsilon}$, or $\|T\|^2 < 1 - t_+ - t_-$. Since $\|T\| \leq \max\{|t_+|, |t_-|\}$, this is implied by the condition given in the Theorem. The result then follows from the cited Theorem. \square

2.5. Positivity of the Fock representation for positive T .

In this section we show a restricted form of monotonicity of the map $T \mapsto P_n(T)$, namely that the positivity of the operator T is also a sufficient condition for the positivity of the Fock representation. We will comment on possible stronger versions of monotonicity at the end of the section.

Theorem 2.5.1. *Let T be a bounded operator on $\mathcal{H} \otimes \mathcal{H}$ with $T \geq 0$, and define $P_n(T)$ as in Lemma 2.1.1. Then $P_{n+1}(T) \geq \mathbb{1} \otimes P_n(T) \geq \mathbb{1}$ for all n .*

Proof. It suffices to consider the finite dimensional case $d < \infty$, since $P_n(pTp)$ for p a finite dimensional projection converges strongly to $P_n(T)$ as $p \nearrow \mathbb{1}$. So let

$$\langle i\ell | T | jk \rangle = \sum_{\alpha} \overline{\Phi_{i\ell}^{\alpha}} \Phi_{jk}^{\alpha} . \quad (2.5.1)$$

We now proceed by induction and assume that $P_n \geq 0$. We have to show that the second term on the right in

$$\begin{aligned} \langle i_0, i_1, \dots, i_n \mid j_0, j_1, \dots, j_n \rangle_T &= \delta_{i_0, j_0} \langle i_1, \dots, i_n \mid j_1, \dots, j_n \rangle_T \\ &\quad + \sum_{k\ell} \langle i_0 \ell | T | j_0 k \rangle \langle i_1, \dots, i_n \mid \lambda_0(\ell k^\dagger) \mid j_1, \dots, j_n \rangle_T \end{aligned}$$

is positive definite. So let $\Psi_{i_0, i_1, \dots, i_n}$ be arbitrary complex coefficients. Then

$$\begin{aligned} \sum \overline{\Psi_{i_0, i_1, \dots, i_n}} \Psi_{j_0, j_1, \dots, j_n} \langle i_0 \ell | T | j_0 k \rangle \langle i_1, \dots, i_n | \lambda_0(\ell k^\dagger) | j_1, \dots, j_n \rangle_T \\ = \sum_{\alpha} \left\| \Psi_{j_0, j_1, \dots, j_n} \Phi_{j_0 k}^{\alpha} \lambda_0(k^\dagger) | j_1, \dots, j_n \rangle \right\|_T^2 \quad . \end{aligned}$$

This norm is evaluated in the n -particle Fock space, whose scalar product is positive definite by inductive hypothesis. Hence $P_{n+1} = \mathbb{I} \otimes P_n + \text{positive terms}$. The bound $P_n \geq \mathbb{I}$ follows from this by induction, since $P_1 = \mathbb{I}$. \square

In view of Theorem 2.3.2 one might be led to believe that the above Theorem can be improved to hold for all $T \geq -(1/2)\mathbb{I}$. However, the following example, a slight modification of Example 2.3.3 shows that the lower bound $T \geq 0$ in the Theorem is optimal:

Example 2.5.2. Consider the Wick algebra $\mathcal{W}(T)$ on two generators x, y with the relations

$$\begin{aligned} x^\dagger y &= y^\dagger x = 0 \\ x^\dagger x &= \mathbb{I} + \lambda x x^\dagger + \varepsilon y y^\dagger \\ y^\dagger y &= \mathbb{I} + \varepsilon x x^\dagger + \lambda y y^\dagger \quad , \end{aligned} \tag{2.5.2}$$

where $\lambda > \varepsilon$. Then ε is the best lower bound for T , and, for any $\varepsilon < 0$, we can find λ such that the Fock representation of $\mathcal{W}(T)$ is not positive.

Proof. Here the coefficients are

$$T |ij\rangle = (\lambda \delta_{ij} + \varepsilon(1 - \delta_{ij})) |ij\rangle \quad .$$

Then $\mathbb{I} + T \geq (1 + \varepsilon)\mathbb{I} > 0$ is boundedly invertible, and $P_3 = \mathbb{I} + T_2 + (\mathbb{I} + T_2)T_1(\mathbb{I} + T_2)$ is positive iff $(\mathbb{I} + T_2)^{-1} + T_1$ is positive. But

$$\langle xyy | (\mathbb{I} + T_2)^{-1} + T_1 | xyy \rangle = \frac{1}{1 + \lambda} + \varepsilon \quad .$$

Hence for any $\varepsilon < 0$ it suffices to take $\lambda > 1 + (-\varepsilon)^{-1}$ to obtain a relation with $T \geq \varepsilon\mathbb{I}$ and non-positive Fock space. \square

Note that this example also disproves the monotonicity conjecture that $(T' \leq T)$ implies $(P_n(T') \leq P_n(T))$: for $T' = -1$, i.e. Example 1.2.7, we have $P_n(T') \geq 0$ for all n , but in the example above we have $T \geq \varepsilon\mathbb{I} > T'$, as long as $\varepsilon \geq -1$, and $P_n(T)$ is not positive.

2.6. T satisfying the braid relations.

We have seen that without further assumptions the condition $\|T\| \leq 1/2$ is the optimal sufficient bound for the Fock representation to be positive. In Example 1.2.1 and Example 1.2.3, however, it is the bound $\|T\| \leq 1$ that marks the boundary of Fock positivity. A similar observation was made by Speicher [Sp2], who generalized the q -relations to the “ q_{ij} ”-relations

$$i^\dagger j = \delta_{ij} \mathbb{1} + q_{ij} j i^\dagger, \quad (2.6.1)$$

where the q_{ij} are real constants. Then

$$T|ij\rangle = q_{ij}|ji\rangle, \quad (2.6.2)$$

and this operator has a natural factorization into $T = Q\pi_1$, where

$$Q|ij\rangle = q_{ji}|ij\rangle, \quad (2.6.3)$$

and $\pi_1|ij\rangle = |ji\rangle$ is the flip operator. Then $\|T\| = \|Q\| = \max_{ij} |q_{ij}|$, and Speicher’s result [Sp2] is again the Fock positivity for $\|T\| \leq 1$.

One feature that makes these relations tractable is the commutativity of Q with all its permuted versions. In fact, the following Proposition shows that this feature can be used to characterize relations of the form (2.6.1). We use the following notation: when \mathcal{H} is a Hilbert space, and Q is a bounded operator on $\mathcal{H} \otimes \mathcal{H}$, we denote by Q_{ij} the operator on the n^{th} tensor power of \mathcal{H} , acting on the tensor slots i, j as Q acts on slots 1, 2, or, more precisely, Q_{ij} is the image of $Q \otimes \mathbb{1}^{\otimes(n-2)}$ under the permutation automorphism taking $(1, 2)$ to (i, j) .

Proposition 2.6.1. *Let \mathcal{H} be a finite dimensional Hilbert space, and let Q be an operator on $\mathcal{H} \otimes \mathcal{H}$. Then the following are equivalent:*

- (1) $Q_{12} = Q_{21}^*$, and the operators Q_{ij} and $Q_{k\ell}$ commute for all n , and $i, j, k, \ell = 1, \dots, n$.
- (2) There are a basis $|i\rangle$ in \mathcal{H} , and complex numbers q_{ij} such that $Q|ij\rangle = q_{ij}|ij\rangle$.

Proof. Let $\mathbb{E}_1^\omega : \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ denote the conditional expectation with respect to a functional ω on the second factor, i.e. $\rho(\mathbb{E}_1^\omega(A \otimes B)) = \rho \otimes \omega(A \otimes B)$, and define \mathbb{E}_2^ω similarly with respect to a functional on the first factor. Then because Q_{12} and Q_{13} commute, we find that the operators of the form $\mathbb{E}_1^\omega(Q)$ commute, for all choices of ω . Similarly, the operators $\mathbb{E}_2^\omega(Q)$ commute, and because $Q_{12} = Q_{21}^*$ we find among them the adjoints of the first commuting set. Finally, because Q_{12} and Q_{23} commute, the $\mathbb{E}_i^\omega(Q)$, $i = 1, 2$, also commute with each other, and hence generate an abelian $*$ -subalgebra $\mathcal{Z} \subset \mathcal{B}(\mathcal{H})$. Let P_α denote the minimal projections of \mathcal{Z} . Then because $\mathbb{E}_1^\omega(Q) \in \mathcal{Z}$ we must have $\mathbb{E}_1^\omega(Q) = \sum_\alpha \omega(X_\alpha) P_\alpha$, and by the symmetrical condition, $X_\alpha \in \mathcal{Z}$. Hence we have $Q \in \mathcal{Z} \otimes \mathcal{Z}$, or $Q = \sum_{\alpha\beta} \tilde{q}_{\alpha\beta} P_\alpha \otimes P_\beta$. The result now follows by choosing a basis in \mathcal{H} in which \mathcal{Z} is diagonal. \square

Note that the first condition in 2.6.1(1) is just the hermiticity of T . Moreover, the Proposition shows that it is natural to consider complex q_{ij} rather than just real ones. The following Proposition shows that in this case, too, $\|T\| \leq 1$ implies Fock positivity. Since $T \geq -\mathbb{1}$ is equivalent to the positivity of P_2 , this bound is clearly optimal.

Proposition 2.6.2. *For $i, j = 1, \dots, d$, let $q_{ij} \in \mathbb{C}$ with $\overline{q_{ij}} = q_{ji}$, and $|q_{ij}| \leq 1$. Then the Wick algebra defined by $i^\dagger j = \delta_{ij} \mathbb{1} + q_{ij} j i^\dagger$ has a positive Fock representation.*

We proved this result in an earlier version of this paper. In the meantime, however, Bożejko and Speicher [BS2] have shown a strictly stronger result, which has a very natural statement in the setting of Wick algebras. We will therefore describe their result below, and give only a sketch of our direct proof of the above Proposition. In the course of this sketch we will also describe the structures needed to state the result of [BS2].

Consider for the moment the factorization $T = Q\pi_1$ without further assumptions on Q . Then

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \iff Q_{ij} Q_{ik} Q_{jk} = Q_{jk} Q_{ik} Q_{ij} \quad , \quad (2.6.4)$$

where we have taken into account that by applying permutation automorphisms to the equation on the right, the special case $(i, j, k) = (i, i+1, i+2)$ is transformed into the case of general (i, j, k) . (In fact, (2.6.4) is nothing but the equivalence of the braid relation stated in terms of the operations T_i of braiding the current i^{th} and $(i+1)^{\text{th}}$ strands, and the same relation stated in terms of the operations Q_{ij} braiding the i^{th} and j^{th} labelled strand). Under the assumptions of Proposition 2.6.2, Lemma 2.6.1(1) clearly implies the validity of the identity on the right, and hence the T_i satisfy the braid relation on the left. This has an important consequence. Consider the inductive expression for P_n in Lemma 2.1.1. Multiplying out the product for P_n we obtain $n!$ terms. Each one of these terms is labelled in a unique way by a permutation π , which is obtained by replacing the product $T(\pi) = T_{i_1} T_{i_2} \cdots T_{i_k}$ of T -operators by $\pi = \pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$, where π_i is the transposition exchanging i and $i+1$. Moreover, $\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ is one of the representations of π as a minimal-length product of transpositions. Now the braid relation (2.6.4) has the consequence that different minimal-length factorizations of π lead to the same operator $T(\pi)$. To see this, represent each permutation by a planar diagram with points labelled $1, \dots, n$ on the bottom and top lines. Now connect each point i in the top row with $\pi(i)$ in the bottom line, such that no connecting line turns upwards anywhere, and such that no three lines intersect in the same point. This diagram can be read as a representation of π as a product of nearest neighbour transpositions. The product is of minimal length iff no two lines intersect twice. Now if the braid relation holds, we can move one strand in this diagram over the crossing of any other two, and thus we can transform any two ways of connecting the top and bottom rows according to π into each other. Hence the braid relation implies that $T(\pi)$ depends only on the permutation π , and is “quasi-multiplicative” [BS2], i.e. $T(\pi\sigma) = T(\pi)T(\sigma)$, as long as π and σ are sub-products in a minimal length factorization of $\pi\sigma$.

The crucial property to establish to get Proposition 2.6.2 and its generalizations is the positive definiteness of the kernel $(\pi, \sigma) \mapsto T(\pi^{-1}\sigma)$, i.e. the operator inequality

$$\sum_{\pi, \sigma} \overline{a_\pi} a_\sigma T(\pi^{-1}\sigma) \geq 0 \quad (2.6.5)$$

for arbitrary choices of $a_\pi \in \mathbb{C}$. For if this holds, we get $P_n(T) = \sum_\pi T(\pi) = (1/N!) \sum_{\pi, \sigma} T(\pi^{-1}\sigma) \geq 0$.

Sketch of proof of 2.6.2: We want to use the commutativity of the Q_{ij} . Hence, for every permutation π , we introduce $Q(\pi) = T(\pi)\pi^{-1}$. Then, because

$$\pi Q_{ij} = Q_{\pi i, \pi j} \pi \quad ,$$

we find that $Q(\pi)$ is a product of factors Q_{ij} . This is true even when these factors do not commute. However, if they do, we get the following simple expression for $Q(\pi)$:

$$Q(\pi) = \prod_{\substack{i < j \\ \pi^{-1}i > \pi^{-1}j}} Q_{ij} \quad .$$

Consider the kernel

$$\widehat{Q}(\pi, \sigma) := \pi T(\pi^{-1}\sigma)\sigma^{-1} = \pi Q(\pi^{-1}\sigma)\pi^{-1} = \prod_{\substack{\pi^{-1}i < \pi^{-1}j \\ \sigma^{-1}i > \sigma^{-1}j}} Q_{ij} \quad .$$

Then the positivity of this kernel is equivalent to the positivity of the kernel $T(\pi^{-1}\sigma)$. Moreover, \widehat{Q} is an abelian product of simpler kernels, each one associated with a pair $(i, j), (j, i)$ of indices. These kernels are readily seen to be positive definite, and hence so is \widehat{Q} . \square

It is clear from this outline that the braid property of T , as the necessary condition for defining the operators $T(\pi)$, is a crucial element of this proof. On the other hand, in Example 1.2.3 the braid property is also satisfied. This had suggested to us the question whether the braid property alone suffices to show Fock positivity up to $\|T\| = 1$. The affirmative answer was given by [BS2] in the following Theorem, which we cite here in the terminology of this paper.

Theorem ([BS2]) 2.6.3. *Let T be a bounded operator on $\mathcal{H} \otimes \mathcal{H}$, satisfying the braid relation $T_1 T_2 T_1 = T_2 T_1 T_2$, and define the operators $P_n(T)$ as in Lemma 2.1.1. Then $\|T\| \leq 1$, implies $P_n(T) \geq 0$ for all n , and $\|T\| < 1$ implies $P_n(T) > 0$ for all n .*

We have already argued that the first bound is optimal. To see the optimality of the second, consider the CAR-relations.

3. IDEALS IN WICK ALGEBRAS

This section deals with an important special kind of ideals in a Wick algebra, which we will call “Wick ideals”. Loosely speaking, they are the ideals which are generated by subsets of $\mathcal{T}(\mathcal{H})$, i.e. by functions of the generators $i \in I$ which do not depend on the adjoints i^\dagger . Their study is motivated by the well-known case of the canonical (anti-) commutation relations, which are usually given as *two* sets of relations: one set of Wick ordering type, for commuting a_i with a_j^* , and another set for commuting a_i with a_j . Wick ideals describe those additional relations that can be introduced consistently in a given Wick algebra.

The consistency problem between Wick relations and the relations defining a Wick ideal has often been posed in the literature from the opposite perspective [WZ,PW,Man]. These authors usually start from an algebra (without involution) defined by generators and relations. They then ask for the possible choices of coefficients $T_{ij}^{k\ell}$ such that the corresponding rules (1.1.1) are consistent with the already given relations in the algebra. In order to put this more formally, recall that in $\mathcal{W}(T)$ there are no relations between the generators i , so that the canonical embeddings $\mathcal{T}(\mathcal{H}) \hookrightarrow \mathcal{W}(T)$ and $\mathcal{T}(\mathcal{H}^\dagger) \hookrightarrow \mathcal{W}(T)$ are injective. However, this injectivity is not essential to the idea of Wick ordering. A more general approach would be to consider a $*$ -algebra \mathcal{W} , which is generated by a subalgebra $\mathcal{A} \subset \mathcal{W}$, and its adjoints $\mathcal{A}^\dagger = \{x^\dagger \mid x \in \mathcal{A}\}$ in such a way that factors from \mathcal{A} and \mathcal{A}^\dagger may be disentangled completely. In other words, we demand that \mathcal{W} is spanned linearly by $\mathcal{A}\mathcal{A}^\dagger$. If \mathcal{A} is generated by $i \in I$, we have a homomorphism $\mathcal{T}(\mathcal{H}) \rightarrow \mathcal{A} \hookrightarrow \mathcal{W}$, which is now no longer injective. Its kernel is the ideal describing the relations between generators. It is still clear that there must be some rule reordering $i^\dagger j$ into products $k^\dagger \ell$, but this commutation rule may not be chosen independently of the ideal. In this picture “Wick ideals” are the ideals in $\mathcal{T}(\mathcal{H})$ which are consistent with the commutation rules given by T .

The importance of Wick ideals for the questions of positivity and boundedness addressed in this paper is twofold: on the one hand it is easy to see that certain ideals of this kind are annihilated by certain coherent representations (see Lemma 3.1.2 below). For example, every quadratic ideal (in the sense described below) automatically vanishes in the Fock representation. The additional relations allow one to simplify the representation space, and so to decide positivity questions more easily. An extreme example are the twisted commutation relations of Example 1.2.3, where the verification of the positivity of the Fock representation reduces to a triviality [PW,Pus]. On the other hand, we may consider a Wick ideal, and study its representation theory as an algebra in its own right. We will find that the generators of a quadratic Wick ideal satisfy “Wick relations without constant term”. These have typically no bounded representations, so that we can conclude that the ideal has to vanish in every bounded representation of the given Wick algebra. We will treat Example 1.2.3 from this point of view in Section 3.2.

3.1. Wick ideals and quadratic Wick ideals.

In the following Lemma we use the following standard notation: when X, Y are linear subspaces of an algebra, XY denotes the linear span of the products xy with $x \in X$, $y \in Y$. With this notation, the characteristic property of Wick algebras is

$$\begin{aligned} \mathcal{H}^\dagger \mathcal{H} &\subset \mathbb{C}\mathbb{I} + \mathcal{H}\mathcal{H}^\dagger \\ \mathcal{T}(\mathcal{H}^\dagger)\mathcal{T}(\mathcal{H}) &\subset \mathcal{T}(\mathcal{H})\mathcal{T}(\mathcal{H}^\dagger) = \mathcal{W}(T) \quad . \end{aligned} \tag{3.1.1}$$

Lemma 3.1.1. *Let $\mathcal{W}(T)$ be a Wick algebra $\mathcal{J} \subset \mathcal{T}(\mathcal{H})$ be an ideal in the tensor algebra over \mathcal{H} , and let $\mathcal{J}_0 \subset \mathcal{J}$ be a subset generating \mathcal{J} . Then the following conditions are equivalent:*

- (1) $\mathcal{J}\mathcal{T}(\mathcal{H}^\dagger)$ is a two-sided ideal in $\mathcal{W}(T)$.
- (2) $\mathcal{T}(\mathcal{H}^\dagger)\mathcal{J} \subset \mathcal{J}\mathcal{T}(\mathcal{H}^\dagger)$.
- (3) $\mathcal{H}^\dagger \mathcal{J}_0 \subset \mathcal{J} + \mathcal{J}\mathcal{H}^\dagger$.

If these conditions are satisfied, \mathcal{J} is called a **Wick ideal**.

Proof. (1) \iff (2): (1) trivially implies (2). For the converse note that the set $\mathcal{J}\mathcal{T}(\mathcal{H}^\dagger)$ is clearly invariant under multiplication with elements of $\mathcal{T}(\mathcal{H}^\dagger)$ from the right and $\mathcal{T}(\mathcal{H})$ from the left. By Wick ordering we have $\mathcal{T}(\mathcal{H}^\dagger)\mathcal{T}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})\mathcal{T}(\mathcal{H}^\dagger)$, hence

$$\mathcal{J}\mathcal{T}(\mathcal{H}^\dagger)\mathcal{T}(\mathcal{H}) \subset \mathcal{J}\mathcal{T}(\mathcal{H})\mathcal{T}(\mathcal{H}^\dagger) \subset \mathcal{J}\mathcal{T}(\mathcal{H}^\dagger) \quad .$$

The only property missing to make $\mathcal{J}\mathcal{T}(\mathcal{H}^\dagger)$ an ideal is the invariance with respect to multiplication by $\mathcal{T}(\mathcal{H}^\dagger)$ from the left, which is (2).

Suppose (3) holds. Then since $\mathcal{J} = \mathcal{H}\mathcal{J}_0\mathcal{H}$ we get $\mathcal{H}^\dagger \mathcal{J} \subset (\mathbb{C}\mathbb{I} + \mathcal{H}\mathcal{H}^\dagger)\mathcal{J}_0\mathcal{H} \subset \mathcal{J} + \mathcal{H}(\mathcal{J} + \mathcal{J}\mathcal{H}^\dagger)\mathcal{H} \subset \mathcal{J} + \mathcal{J}(\mathbb{C}\mathbb{I} + \mathcal{H}\mathcal{H}^\dagger) \subset \mathcal{J} + \mathcal{J}\mathcal{H}^\dagger$. Hence (3) is equivalent to (3) with \mathcal{J}_0 replaced by \mathcal{J} .

(2) \iff (3): The right hand side of (3) is contained in $\mathcal{J}\mathcal{T}(\mathcal{H}^\dagger)$. Hence (3) implies (2) by induction on the degree of monomials in $\mathcal{T}(\mathcal{H}^\dagger)$. Conversely, if (2) holds, we have, on the one hand $\mathcal{H}^\dagger \mathcal{J} \subset \mathcal{T}(\mathcal{H}) + \mathcal{T}(\mathcal{H})\mathcal{H}^\dagger$ by Wick ordering, and, on the other hand $\mathcal{H}^\dagger \mathcal{J} \subset \mathcal{J}\mathcal{T}(\mathcal{H}^\dagger) = \mathcal{J} + \mathcal{J}\mathcal{H}^\dagger + \mathcal{J}\mathcal{H}^\dagger\mathcal{H}^\dagger + \dots$. Now each term appearing on the right hand side of these expressions is Wick ordered. Since the Wick ordered polynomials form a linear basis of $\mathcal{W}(T)$, we may compare coefficients of monomials in adjoint generators i^\dagger , and obtain $\mathcal{H}^\dagger \mathcal{J} \subset \mathcal{J} + \mathcal{J}\mathcal{H}^\dagger$. \square

Note that the Lemma does not assert that $\mathcal{J}\mathcal{T}(\mathcal{H}^\dagger)$ is a star-ideal in $\mathcal{W}(T)$. Hence in order to lift the involution of $\mathcal{W}(T)$ to the quotient we have to divide out the larger ideal $\mathcal{J}\mathcal{T}(\mathcal{H}^\dagger) + \mathcal{T}(\mathcal{H})\mathcal{J}^\dagger$. The following Lemma gives a simple criterion for deciding which coherent representations lift to the quotient.

Lemma 3.1.2. *Let \mathcal{J} be a Wick ideal in a Wick algebra $\mathcal{W}(T)$, and let $\varphi : \mathcal{H} \rightarrow \mathbb{C} : f \mapsto \langle f, \varphi \rangle$ be a conjugate linear functional. Denote by $\widehat{\varphi} : \mathcal{T}(\mathcal{H}) \rightarrow \mathbb{C}$ the induced homomorphism with $\widehat{\varphi}(\mathbb{1}) = 1$, and $\widehat{\varphi}(f) = \langle \varphi, f \rangle = \overline{\langle f, \varphi \rangle}$, and denote by $\widehat{\lambda}_\varphi$ the separated coherent representation associated with φ (cf. Definition 1.3.2). Then $\widehat{\lambda}_\varphi(\mathcal{J}) = \{0\}$ if and only if $\widehat{\varphi}(\mathcal{J}) = \{0\}$.*

Proof. By definition of the coherent representation λ_φ , we have, for all $G \in \mathcal{T}(\mathcal{H})$, and $f \in \mathcal{H}$: $\langle \mathbb{1}, \lambda_\varphi(f) \lambda_\varphi(G) \mathbb{1} \rangle_{T, \varphi} = \langle \lambda_\varphi(f^\dagger) \mathbb{1}, \lambda_\varphi(G) \mathbb{1} \rangle_{T, \varphi} = \overline{\langle f, \varphi \rangle} \langle \mathbb{1}, \lambda_\varphi(G) \mathbb{1} \rangle_{T, \varphi}$. By induction on the degree of F , we have

$$\left\langle \mathbb{1}, \lambda_\varphi(F) \lambda_\varphi(G) \mathbb{1} \right\rangle_{T, \varphi} = \widehat{\varphi}(F) \left\langle \mathbb{1}, \lambda_\varphi(G) \mathbb{1} \right\rangle_{T, \varphi} .$$

In particular, for $G = \mathbb{1}$, we get $\langle \mathbb{1}, \widehat{\lambda}_\varphi(F) \mathbb{1} \rangle_{T, \varphi} = \widehat{\varphi}(F)$. Hence $\widehat{\lambda}_\varphi(\mathcal{J}) = \{0\}$ implies $\widehat{\varphi}(\mathcal{J}) = \{0\}$. Conversely, if $\widehat{\varphi}(\mathcal{J}) = \{0\}$, we have $\langle \mathbb{1}, \widehat{\lambda}_\varphi(\mathcal{J}) \widehat{\lambda}_\varphi(\mathcal{T}(\mathcal{H})) \mathbb{1} \rangle_{T, \varphi} = \{0\}$, and, by the Wick ideal property:

$$\begin{aligned} \left\langle \widehat{\lambda}_\varphi(\mathcal{T}(\mathcal{H})) \mathbb{1}, \widehat{\lambda}_\varphi(\mathcal{J}) \widehat{\lambda}_\varphi(\mathcal{T}(\mathcal{H})) \mathbb{1} \right\rangle_{T, \varphi} &= \left\langle \mathbb{1}, \widehat{\lambda}_\varphi(\mathcal{T}(\mathcal{H}^\dagger) \mathcal{J}) \mathbb{1} \right\rangle_{T, \varphi} \\ &\subset \left\langle \mathbb{1}, \widehat{\lambda}_\varphi(\mathcal{J}) \widehat{\lambda}_\varphi(\mathcal{T}(\mathcal{H}^\dagger)) \mathbb{1} \right\rangle_{T, \varphi} = \{0\} \end{aligned}$$

Since the vectors $\widehat{\lambda}_\varphi(\mathcal{T}(\mathcal{H})) \mathbb{1}$ span the representation space of $\widehat{\lambda}_\varphi$, we get $\widehat{\lambda}_\varphi(\mathcal{J}) = \{0\}$. \square

We now consider ideals which are generated by homogeneous quadratic expressions. The following result, stated in a slightly different setup, can be found in [WZ].

Proposition 3.1.3. *Let $\mathcal{W}(T)$ be a Hermitian Wick algebra. Let P be an orthogonal projection on $\mathcal{H} \otimes \mathcal{H}$, and consider the ideal $\mathcal{J} \subset \mathcal{T}(\mathcal{H})$ generated by $P\mathcal{H} \otimes \mathcal{H}$. Then \mathcal{J} is a Wick ideal if and only if*

$$(1 + T)P = 0 \tag{3.1.2a}$$

$$\text{and} \quad (\mathbb{1} \otimes (\mathbb{1} - P))(T \otimes \mathbb{1})(\mathbb{1} \otimes T)(P \otimes \mathbb{1}) = 0 \quad , \tag{3.1.2b}$$

where T is the operator introduced in equation (2.1.2). Ideals \mathcal{J} of this form will be called **quadratic ideals**.

Proof. The ideal is generated by the elements of the form

$$A_{ij} = \sum_{k\ell} P_{ij}^{k\ell} k \otimes \ell \quad ,$$

where $i, j \in I$. (Typically, these are linearly dependent). A straightforward computation using (1.1.1) gives

$$k^\dagger A_{ij} = \left(P_{ij}^{km} + P_{ij}^{nr} T_{kn}^{rm} \right) m + P_{ij}^{nm} T_{kn}^{rs} T_{rm}^{uv} s v u^\dagger \quad ,$$

where we have used the convention that every $i \in I$ appearing twice is automatically summed over. By Lemma 3.1.1(3) the first term must be in \mathcal{J} , and the second in \mathcal{JH}^\dagger . Since $\mathcal{J} \subset \mathcal{T}(\mathcal{H})$ is generated by homogeneous quadratic expressions, the first term, being only linear, must be zero, and the second must be of the form $J_u u^\dagger$ with $J_u \in \mathcal{J}$ for all u . Using $P_{ij}^{k\ell} = \langle k\ell | P | ij \rangle$, and (2.1.2), the vanishing of the first degree term is written as

$$0 = \langle km | P | ij \rangle + \langle km | T | nr \rangle \langle nr | P | ij \rangle = \langle km | (\mathbb{1} + T) P | ij \rangle \quad ,$$

for all $k, m, i, j \in I$.

Since P was chosen as a projection, we can express the condition $J_u \in \mathcal{J}$ as

$$\begin{aligned} 0 &= P_{ij}^{nm} T_{kn}^{rs} T_{rm}^{uv} (\mathbb{1} - P)_{sv}^{pq} \\ &= \langle pq | \mathbb{1} - P | sv \rangle \langle ks | T | nr \rangle \langle rv | T | mu \rangle \langle nm | P | ij \rangle \\ &= \langle kpq | \mathbb{1} \otimes (\mathbb{1} - P) | k' sv' \rangle \langle k' sv' | T \otimes \mathbb{1} | nr v \rangle \langle nr v | \mathbb{1} \otimes T | n' mu' \rangle \langle n' mu' | P \otimes \mathbb{1} | iju \rangle \\ &= \langle kpq | (\mathbb{1} \otimes (\mathbb{1} - P)) (T \otimes \mathbb{1}) (\mathbb{1} \otimes T) (P \otimes \mathbb{1}) | iju \rangle \quad , \end{aligned}$$

for all $k, p, q, i, j, u \in I$. \square

The two conditions on P and T are called the “linear” and the “quadratic” condition in [WZ]. Note that any quadratic ideal is annihilated by the Fock representation. This gives an immediate interpretation of the linear condition: since $P_2 = \mathbb{1} + T$ determines the scalar product in 2-particle Fock space, the norm of the generators A_{ij} , considered as vectors in Fock space, must vanish. In particular, a necessary condition for $\mathcal{W}(T)$ to have a non-trivial quadratic ideal is that -1 is an eigenvalue of T . As Example 1.2.2 at $qd = -1$ shows, this condition is not sufficient. However, if the braid relation for T holds (compare Example 1.2.5, and Sections 2.6, and 4.3), the quadratic condition does become redundant. We record this simple observation for future reference.

Corollary 3.1.4. *Suppose T satisfies the braid condition $T_1 T_2 T_1 = T_2 T_1 T_2$, and let P be the eigenprojection of T for eigenvalue -1 . Then the ideal in $\mathcal{T}(\mathcal{H})$ generated by $P(\mathcal{H} \otimes \mathcal{H})$ is a quadratic Wick ideal.*

Proof. (3.1.2a) is obviously satisfied. Moreover, $T_2(T_1 T_2 P_1) = (T_1 T_2) T_1 P_1 = -(T_1 T_2 P_1)$. Hence $T_1 T_2 P_1$ maps into the eigenspace of T_2 for eigenvalue -1 , and $(\mathbb{1} - P_2) T_1 T_2 P_1 = 0$, proving (3.1.2a). Now Proposition 3.1.3 applies. \square

Let \mathcal{J} be a Wick ideal. Then $\mathcal{J}^\dagger \mathcal{J} \subset \mathcal{T}(\mathcal{H}^\dagger) \mathcal{J} \subset \mathcal{J} \mathcal{T}(\mathcal{H}^\dagger)$, and by taking adjoints, $\mathcal{J}^\dagger \mathcal{J} \subset \mathcal{T}(\mathcal{H}) \mathcal{J}^\dagger$. This suggests that $\mathcal{J}^\dagger \mathcal{J} \subset \mathcal{J} \mathcal{J}^\dagger$. We do not know whether this is the case in general. However, for quadratic ideals this inclusion holds. More precisely, we get the following relations.

Lemma 3.1.5. *Let T and P be as in Proposition 3.1.3, and set*

$$A_{ij} = P(i \otimes j) = \sum_{k\ell} P_{ij}^{k\ell} k \otimes \ell \quad ,$$

for $i, j \in I$. Then

$$A_{k_1 k_2}^\dagger A_{i_1 i_2} = \sum_{r_1 r_2 s_1 s_2} \langle k_1 k_2 r_1 r_2 | P_1 P_3 T_2 T_1 T_3 T_2 P_3 P_1 | i_1 i_2 s_1 s_2 \rangle A_{r_1 r_2} A_{s_1 s_2}^\dagger \quad ,$$

where as in equation (2.1.5) the indices on T and P indicate the tensor factors in which the respective operators act. In particular, $\mathcal{J}^\dagger \mathcal{J} \subset \mathcal{J} \mathcal{J}^\dagger$, where \mathcal{J} is the associated Wick ideal.

Proof. Following the proof of Proposition 3.1.3 we find

$$\begin{aligned} k^\dagger A_{i_1 i_2} &= \langle k s_1 s_2 | T_1 T_2 P_1 | i_1 i_2 u \rangle A_{s_1 s_2} u^\dagger \\ \ell^\dagger k^\dagger A_{i_1 i_2} &= \langle k \ell r_1 r_2 | T_2 T_3 P_2 T_1 T_2 P_1 | i_1 i_2 uv \rangle A_{r_1 r_2} v^\dagger u^\dagger \\ A_{k_1 k_2}^\dagger A_{i_1 i_2} &= \langle k_1 k_2 r_1 r_2 | P_1 T_2 T_3 P_2 T_1 T_2 P_1 | i_1 i_2 uv \rangle A_{r_1 r_2} v^\dagger u^\dagger \quad . \end{aligned}$$

We now use the relation $T_1 T_2 P_1 = P_2 T_1 T_2 P_1$ from Proposition 3.1.3, together with its translate $T_2 T_3 P_2 = P_3 T_2 T_3 P_2$, and their adjoints, and the commutation property $X_1 X_3 = X_3 X_1$ for $X = P, T$, to get

$$\begin{aligned} P_1 (T_2 T_3 P_2) T_1 T_2 P_1 &= P_1 P_3 T_2 T_3 (P_2 T_1 T_2 P_1) = (P_1 P_3) T_2 (T_3 T_1) T_2 P_1 \\ &= P_3 (P_1 T_2 T_1) T_3 T_2 P_1 = (P_1 P_3) T_2 T_1 (P_2 T_3 T_2) P_1 \\ &= P_3 P_1 T_2 T_1 T_3 T_2 P_3 P_1 \quad , \end{aligned}$$

where the brackets indicate to which part of the product a rule is applied in the next step. Then the result follows by introducing the definition of $A_{s_1 s_2}^\dagger$ \square

Hence the generators of a quadratic Wick ideal satisfy *Wick commutation relations without constant term*. When T satisfies the braid condition, the expression $T_2 T_1 T_3 T_2$, which determines the commutation relations of the A_{ij} , has a simple intuitive interpretation: it describes the braiding operation on four strands, consisting of taking the strands two and two together, and braiding these “cables” once. In particular, the coefficients for the algebra $\mathcal{J} \mathcal{J}^\dagger$ themselves satisfy the braid condition.

Wick relations without constant term behave quite differently from the Wick relations (1.1.1). Being homogeneous in the generators, they always have $A_{ij} = 0$ as a solution. On the other hand, if there is any non-trivial bounded solution, any multiple is also a solution, so there cannot be a universal bounded representation. For the existence of bounded representations it is instructive to consider the case of a single generator, i.e. $x^\dagger x = \lambda x x^\dagger$.

Since the spectra of xx^\dagger and $x^\dagger x$ are the same (apart from zero), the spectrum of $x^\dagger x$ must be invariant under multiplication by integer powers of λ . Hence either $x = 0$ is the only bounded representation, or $\lambda = 1$, and x is normal. When we look for bounded representations of a Wick algebra which admits a quadratic ideal, we should therefore look first for the bounded representations of the relations in Lemma 3.1.5. In the following section we will study cases in which the ideal is automatically annihilated in any bounded representation.

We close this section with a brief review of the Wick ideals in the examples of Section 1.2. In **Example 1.2.1**, the q -relations and, more generally, for small T (**Example 1.2.9**) we typically have the isomorphism of the universal bounded representation with the Cuntz-Toeplitz algebra. Thus an irreducible representation is either the Fock representation or a representation of the Cuntz algebra, which is simple, and hence tolerates no non-trivial ideals. Hence in these examples we do not have any Wick ideals. Quadratic ideals cannot exist while $T > -\mathbb{I}$.

In **Example 1.2.2**, the Temperley-Lieb-Wick relations, we can choose $q = -d^{-1}$, so that -1 is an eigenvalue of T , and the linear condition (3.1.2a) is satisfied. However, the quadratic condition (3.1.2b) fails, so these two are independent.

For the twisted canonical commutation relations (**Example 1.2.3**) we have a nested sequence of quadratic Wick ideals \mathcal{J}_n , generated by $(ij - \mu ji)$, for $i, j \leq n$. In the fermionic case \mathcal{J}_n is generated by $(ij + \mu ji)$, and i^2 for $i, j \leq n$. Wick ideals of higher order, which are contained in \mathcal{J}_d will also occur in the following section.

In **Example 1.2.6**, the Clifford algebras, we saw that the elements $\theta_{ij} := (ij + ji)$ are central. Hence for any choice of a symmetric, \mathbb{C} -valued matrix $\hat{\theta}_{ij}$, any set of elements $ij + ji - \hat{\theta}_{ij}\mathbb{I}$ defines a Wick ideal. Except for $\hat{\theta} = 0$ this ideal is not quadratic, and since the generators are not of homogeneous degree, the grading loses its meaning in the quotient algebra, and becomes replaced by a \mathbb{Z}_2 -grading.

Finally, **Example 1.2.7** was constructed in order to get the largest possible quadratic Wick ideal, containing all monomials of degree two and higher.

3.2. Twisted canonical (anti-)commutation relations.

Of the examples from Section 1.2 discussed above, the twisted canonical (anti-)commutation relations (**Example 1.2.3**) have the most complex Wick ideal structure. In this subsection we will use this structure to obtain information about the representation theory of these relations. Since T satisfies the braid relations [Bae] we can consider, by Corollary 3.1.4, the largest quadratic Wick ideal, given by the “ -1 ”-eigenspace of T . In the Bosonic case this ideal is generated by all $ij - \mu ji$ for $i > j$, and in the Fermionic case by all i^2 , and all $ij + \mu ji$ for $i > j$. Pusz and Woronowicz have determined [PW,Pus] all

irreducible representations of the relations for which this ideal is annihilated. Therefore, we focus on what can be said without making that assumption.

We consider the Bosonic case first. The first part of the following Theorem is based on the idea of considering the representations of the maximal quadratic Wick ideal as an algebra in its own right. As is often the case for the representation of Wick algebra relations without constant term, we find that only the zero representation is bounded. The further classification of bounded representations then proceeds along the lines set by Pus̄z and Woronowicz [PW] in their classification of the representations in which the Wick ideal is annihilated. Using the notion of coherent representations their classification can be made more transparent, which helped us to detect an omission in their classification (in the notation of the Theorem they have only $\alpha = 1$). Therefore we include a brief, corrected sketch of the classification of the bounded representations as it might have been given in [PW]. We emphasize, however, that there are also *unbounded* representations of the relations, which are classified in [PW], but which are *not* coherent in our sense.

Theorem 3.2.1. *Let π be a bounded representation of the twisted canonical commutation relations (1.2.3). Then*

$$\pi(ij) = \mu\pi(ji) \quad ,$$

for all $i > j \in I$. Moreover, if π is irreducible, it is coherent, i.e. there are a cyclic unit vector Ω , and $\varphi_i \in \mathbb{C}$ such that $\pi(i^\dagger)\Omega = \varphi_i\Omega$. One has either $\varphi_i = 0$ for all i (the Fock representation), or there is an index k , and a phase $\alpha \in \mathbb{C}$, $|\alpha| = 1$, such that

$$\varphi_i = \begin{cases} \alpha(1 - \mu^2)^{-1/2} & i = k \\ 0 & i \neq k \end{cases} . \quad (3.2.1)$$

Representations with different k or α are inequivalent.

Proof. The standard form of the generators given in Lemma 3.1.5 is $A_{ij} = c(ij - \mu ji)$, with an irrelevant overall factor $c = -\mu/(1 + \mu^2)$, which we will drop in the sequel. A tedious but straightforward computation gives, for $i > j$, the following explicit version of the commutation relations found in Lemma 3.1.5:

$$\begin{aligned} A_{ij}^\dagger A_{ij} &= \mu^6 A_{ij} A_{ij}^\dagger - \mu(1 - \mu^2) \sum_{r < j} A_{ir} A_{ir}^\dagger \\ &\quad - \mu^4(1 - \mu^2) \sum_{j < s < i} A_{sj} A_{sj}^\dagger + \mu^2(1 - \mu^2) \sum_{r < j < s < i} A_{sr} A_{sr}^\dagger \\ &\quad - \mu^4(1 - \mu^2) \sum_{r < j} A_{jr} A_{jr}^\dagger \\ &\quad + (1 - \mu^2)^2(1 + \mu^2) \sum_{r < s < j} A_{sr} A_{sr}^\dagger . \end{aligned}$$

Here we have arranged the terms $A_{nm} A_{nm}^\dagger$ on the right hand side in descending lexicographic order with respect to the pairs (n, m) with $n > m$. Note that only pairs

(n, m) which are smaller than (i, j) in this order contribute to this expression. Since for a bounded operator $A_{ij}^\dagger A_{ij} = \mu^6 A_{ij} A_{ij}^\dagger$ implies $A_{ij} = 0$, the result follows by induction on lexicographic order.

We now give a sketch of the further classification of bounded irreducible representations. Consider the operators $E_i = \pi(ii^\dagger)$. Then, because of the relation just proven, these operators commute. Moreover, $\pi(i^\dagger)E_j = \tau_i(E)_j\pi(i^\dagger)$, where $\tau_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an affine transformation:

$$\tau_i(E)_j = \begin{cases} E_j & i < j \\ 1 + \mu^2 E_i - (1 - \mu^2) \sum_{k < i} E_k & i = j \\ \mu^2 E_j & i > j \end{cases}.$$

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function with the property $f \circ \tau_i = f$ for all i . Then $f(E)$ (evaluated in the joint functional calculus of the E_i) commutes with all generators and their adjoints, hence is a multiple of the identity. It follows that the joint spectrum $\sigma(E) \subset \mathbb{R}^d$ of the E_i consists of a single orbit of the τ_i . Let $\lambda \in \sigma(E)$, and let Φ be a vector in the corresponding eigenspace: $E_i \Phi = \lambda_i \Phi$. Then $\pi(i^\dagger)^n \Phi$ is either zero or an eigenvector for the eigenvalues $\tau_i^{-n}(\lambda)$.

Consider the iteration of τ_1^{-1} . Then, as a first case, we may have $\tau_1(\lambda) = \lambda$, i.e. $\lambda_1 = (1 - \mu^2)^{-1}$, and $\lambda_i = 0$ for $i > 1$. If λ is not a fixed point of τ_1 , the iteration of τ_1^{-1} produces an unbounded sequence, hence, eventually, we must have $\pi(1^\dagger)^n \Phi = 0$ (Here “1” denotes the first generator). We may now replace Φ by the last non-zero vector $\pi(1^\dagger)^n \Phi$, and repeat the argument for the iteration of τ_2 , with the additional information that $\lambda_1 = 0$. By iteration we obtain either an eigenvector Φ of E such that $\lambda_i = 0$ for all i , which clearly is a Fock vector. Or, otherwise, we find k , and an eigenvector Φ of E such that $\lambda_k = (1 - \mu^2)^{-1}$, and $\lambda_i = 0$ for $i \neq k$.

Let \mathcal{N} be the subspace of eigenvectors Φ of the latter type. Then since λ is a fixed point of τ_k , we have that $\pi(k)\mathcal{N} \subset \mathcal{N}$, and $\pi(k^\dagger)\mathcal{N} \subset \mathcal{N}$. Moreover, the restriction of $\pi(k)$ to \mathcal{N} is unitary up to the factor $(1 - \mu^2)^{1/2}$. Let U be a unitary operator in \mathcal{N} commuting with the restriction of $\pi(k)$. We extend U to an operator on the cyclic subspace generated by \mathcal{N} , i.e. the whole representation space, by setting

$$U\pi(X)\Phi = \pi(X)U\Phi \quad ,$$

for $\Phi \in \mathcal{N}$, and $X \in \mathcal{W}(T)$. We claim that U is well-defined, and unitary. In $\langle \pi(X_1)U\Phi_1, \pi(X_2)U\Phi_2 \rangle$ we can Wick order the expression $X_1^\dagger X_2$. From the Wick ordered form of $X_1^\dagger X_2$ we can drop all terms containing generators $i \neq k$, because of the commutation relations between the $\pi(i)$, and because $\pi(i^\dagger)\Phi_\nu = 0$, $\nu = 1, 2$. This leaves a polynomial in $\pi(k)$ and $\pi(k^\dagger)$, which commutes with U . Hence $\langle \pi(X_1)U\Phi_1, \pi(X_2)U\Phi_2 \rangle = \langle \pi(X_1)\Phi_1, \pi(X_2)\Phi_2 \rangle$, from which it is clear that U is unitary. Moreover, by definition, U commutes with all $\pi(X)$, and must be a multiple of the identity by irreducibility. It follows that \mathcal{N} is one-dimensional, and hence that $\pi(k)\Phi = z\Phi$, for some $z \in \mathbb{C}$, and $\Phi \in \mathcal{N}$. From

$\pi(kk^\dagger)\Phi = \lambda_k\Phi = (1 - \mu^2)^{-1}\Phi$ we get $z = \alpha(1 - \mu^2)^{-1/2}$, and any unit vector in \mathcal{N} is the desired coherent vector. The inequivalence of the representations with different k or α is evident from this construction. \square

In the above proof we have not established the positivity of the coherent representations. This can be done in a simple way, which gives additional insight into this structure: let π' denote the Fock representation of the relations with $k - 1$ generators, with Fock vector Ω . Let N denote the number operator of this representation, defined by $N\pi'(i_1i_2, \dots, i_n)\Omega = n\pi'(i_1i_2, \dots, i_n)\Omega$. Then we set

$$\pi(i) = \begin{cases} \pi'(i) & i < k \\ \alpha(1 - \mu^2)^{-1/2} \mu^N & i = k \\ 0 & i > k \end{cases} . \quad (3.2.2)$$

The relations for $\pi(ij)$, and $\pi(ij^\dagger)$, with $i \neq j$ readily follow from the definition of the number operator. The only non-trivial relation to be checked is for $\pi(kk^\dagger)$, which reduces to the identity

$$\sum_{i < k} \pi(ii^\dagger) = \frac{1 - \mu^{2N}}{1 - \mu^2} , \quad (3.2.3)$$

which was shown in [PW], equations 2.34 and 2.35. Clearly, this representation is positive, because the Fock representation π' is positive, and it is immediately clear that it is the coherent representation with the parameters specified in the Theorem.

The Fermionic case is more involved. Since the first generator will play a special rôle, and “1” is a confusing notation for the first generator, we use the notational convention introduced in equation (1.1.5), and denote the generators in the representation “ a^\dagger ” under consideration by $a_i^\dagger = a^\dagger(i)$, and their adjoints by $a_i = a^\dagger(i^\dagger)$, with $i = 1, \dots, d \leq \infty$. The relations (1.2.4) then become

$$a_i a_j^\dagger = \begin{cases} -\mu a_j^\dagger a_i & \text{for } i \neq j \\ \mathbb{I} - a_i^\dagger a_i - (1 - \mu^2) \sum_{k < i} a_k^\dagger a_k & \text{for } i = j. \end{cases} \quad (3.2.4)$$

We will refer to these relations as the μ CAR. It is clear that each a_i is bounded, and we need not distinguish the abstract Wick algebraic adjoint “ † ” from the operator adjoint * . As in the Bosonic case, we have a nested sequence of quadratic Wick ideals \mathcal{J}_n , but this time generated by i^2 , and $ij + \mu ji$ for $1 \leq i, j \leq n$.

The following Proposition is the basic tool for investigating these nested ideals in a general representation: it allows us to reduce the case that \mathcal{J}_1 is annihilated to the study of a another representation of the μ CAR with one generator less. Applied inductively, it reduces the study of representations in which \mathcal{J}_n vanishes to the study of the μ CAR with n generators less. In particular, the representations in which the maximal quadratic

Wick ideal \mathcal{J}_d is annihilated, i.e. those representations considered by [Pus], are analyzed completely by the following Proposition, and are given as

$$a_i = \underbrace{\begin{pmatrix} -\mu & 0 \\ 0 & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} -\mu & 0 \\ 0 & 1 \end{pmatrix}}_{(i-1) \text{ factors}} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \underbrace{\mathbb{I} \otimes \cdots \otimes \mathbb{I}}_{(d-i) \text{ factors}} . \quad (3.2.5)$$

This is precisely the μ -deformation of the standard technique for analyzing the canonical anticommutation relations with $\mu = 1$ [BR]. In the general case we get a decomposition of an arbitrary representation π of the μ CAR as $\pi = \pi_0 \oplus \pi_1$ with $\pi_0(\mathcal{J}_1) = \{0\}$, or, equivalently $\pi_0(a_1^2) = 0$. A crucial property needed for this decomposition is the normality of a_1^2 , which follows from the μ CAR. Unless it vanishes, a_1 will not be normal. For the general structure theory of (single) operators with normal square, we refer to [RR].

Proposition 3.2.2. *Let a_i , $i = 1, \dots, d$ be operators on a Hilbert space \mathcal{R} satisfying the μ CAR (3.2.4).*

- (1) *Then a_1^2 is normal, and both summands in the direct sum decomposition $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1$, with $\mathcal{R}_0 = \ker a_1^2$, and $\mathcal{R}_1 = \overline{a_1^2 \mathcal{R}}$, are invariant under all a_i, a_i^* .*
- (2) *\mathcal{R}_0 can be decomposed into $\mathcal{R}_0 = \mathbb{C}^2 \otimes \mathcal{R}'_0$ such that*

$$a_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{I}$$

$$a_i = \begin{pmatrix} -\mu & 0 \\ 0 & 1 \end{pmatrix} \otimes \tilde{a}_{i-1} \quad , \quad \text{for } i > 1,$$

where the operators \tilde{a}_i , $i = 1, \dots, d-1$ again satisfy (3.2.4).

Proof. (1) Since $a_1^* a_1 + a_1 a_1^* = \mathbb{I}$, a_1 determines a representation of the Clifford algebra discussed above in Section 3.1, and $N = a_1^2$ is normal. Then for any $j > 1$:

$$N a_j^* = \mu^2 a_j^* N \quad .$$

By the Fuglede-Putnam-Rosenblum Theorem [Rud], $N^* a_j^* = \mu^2 a_j^* N^*$, or $a_j N = \mu^2 N a_j$. Hence the kernel of N is an invariant subspace for all generators and their adjoints. The orthogonal complement of the kernel is then also an invariant subspace.

(2) In the following we may assume that $\mathcal{R}_1 = \{0\}$, and $\mathcal{R}_0 = \mathcal{R}$. Since $a_1^2 = 0$, a_1 and its adjoint generate a copy of the 2×2 matrix algebra, and we can decompose the space as $\mathcal{R} = \mathbb{C}^2 \otimes \mathcal{R}'$, with a_1 of the form given in the Proposition. What we have to show is the form for the a_i , $i > 1$. We can write $a_i = \begin{pmatrix} C_i & B_i \\ D_i & A_i \end{pmatrix}$, with bounded operators A_i, B_i, C_i, D_i on \mathcal{R}' . Then

$$0 = a_i a_1^* + \mu a_1^* a_i = \begin{pmatrix} \mu D_i & C_i + \mu A_i \\ 0 & D_i \end{pmatrix} \quad .$$

Hence $D_i = 0$, and $C_i = -\mu A_i$. The diagonal block matrix components of the Wick relation for $a_i a_i^*$ ($i > 1$) are

$$\begin{aligned} \mu^2(A_i A_i^* + A_i^* A_i) + B_i B_i^* &= \mathbb{I} - (1 - \mu^2) \mu^2 \sum_{1 < k < i} A_k^* A_k - (1 - \mu^2) \mathbb{I} \\ A_i A_i^* + A_i^* A_i + B_i^* B_i &= \mathbb{I} - (1 - \mu^2) \sum_{1 < k < i} (B_k^* B_k + A_k^* A_k) \quad , \\ \text{and} \quad B_i B_i^* - \mu^2 B_i^* B_i &= \mu^2 (1 - \mu^2) \sum_{1 < k < i} B_k^* B_k \quad , \end{aligned}$$

where the third equation is the first minus μ^2 times the second. For $k = 2$ we get $B_2 B_2^* = \mu^2 B_2^* B_2$, hence $B_2 = 0$ by boundedness. Proceeding by induction we find that $B_i = 0$ for all $i > 1$. Hence $a_i = \begin{pmatrix} -\mu A_i & 0 \\ 0 & A_i \end{pmatrix}$. Substituting $B_i = 0$ in the above equation we find that the A_i with $i > 1$ satisfy the Wick relation for $A_i A_i^*$. Since the block matrix for a_i is diagonal it is clear that the relations for $a_i a_j^*$ with $i, j > 1$ are satisfied if and only if the relations for $A_i A_j^*$ hold. \square

It remains to analyze the representations of the type π_1 . As a first step we show that any irreducible representation of this type is coherent, with a vector Ω satisfying $a_1 \Omega = \alpha \Omega \neq 0$, and $a_i \Omega = 0$ for $i > 1$. Combined with the previous Proposition we get the following statement.

Theorem 3.2.3.

- (1) *Every irreducible representation of the μ CAR (3.2.4) is coherent.*
- (2) *For every coherent representations there are an integer r and $\alpha \in \mathbb{C}$ with $|\alpha| \leq \mu^{r-1}/\sqrt{2}$ such that the cyclic vector Ω is characterized by*

$$a_j \Omega = \alpha \delta_{jr} \Omega \quad . \tag{3.2.6}$$

Proof. By Proposition 3.2.2, an irreducible representation has either $a_1^2 = 0$, or a_1^2 non-singular. In the first case the explicit form for \mathcal{R}_0 makes clear that a is irreducible iff \tilde{a} is, and if \tilde{a} is coherent with cyclic vector Ω , then a is coherent with cyclic vector $\tilde{\Omega} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \Omega$.

Moreover, $\tilde{a}_i \tilde{\Omega} = \tilde{\alpha}_i \tilde{\Omega}$ implies $a_i \Omega = \alpha_i \Omega$ with $\alpha_1 = 0$, and $\alpha_i = -\mu \tilde{\alpha}_{i-1}$ for $i > 1$. Hence, by induction on the number of generators, we may assume in both parts of the Theorem that $\mathcal{R}_0 = \{0\}$, i.e. that $N = a_1^2$ is non-singular, and hence $r = 1$.

We have seen in the proof of Proposition 3.2.2 that $a_j N = \mu^2 N a_j$, and $a_j N^* = \mu^2 N^* a_j$. Hence by the functional calculus

$$a_j f(N) = f(\mu^2 N) a_j \quad ,$$

where $j > 1$, and f is any measurable real-valued function on the complex plane. In particular, let f be a real-valued function with the scaling invariance property $f(x) = f(\mu^2 x)$, and set $F = f(N)$. Then F commutes with all generators, including a_1 . Hence, in an irreducible representation, it must be a multiple of the identity. It follows that the spectrum of N must be contained in just one of the sets $\{\mu^{2n}\alpha^2 \mid n \in \mathbb{Z}\}$, where $\alpha \in \mathbb{C}$. Since N is bounded, we can take α^2 as the element of largest modulus, and obtain a spectrum contained in $\{\mu^{2n}\alpha^2 \mid n \geq 0\}$.

Let φ be in the eigenspace \mathcal{N} of N for the largest eigenvalue α^2 . Then $N^*\varphi = \bar{\alpha}^*\varphi$, and, for $j > 1$: $N^*a_j\varphi = \mu^{-2}\bar{\alpha}^2 a_j\varphi$. Since $a_j\varphi$ cannot be eigenvector of N for an eigenvalue larger than $\|N\|$, we must have $a_j\varphi = 0$. Since \mathcal{N} is also invariant under a_1 , either $(a_1 - \alpha)\mathcal{N}$ or $(a_1 + \alpha)\mathcal{N}$ is non-zero, hence \mathcal{N} contains an eigenvector of a_1 with eigenvalue $\pm\alpha$. Renaming $\alpha \mapsto -\alpha$, if necessary, we obtain the vector described in (2). The bound on α comes from the representation theory of the Clifford algebra (see Example 1.2.6, and [JW]).

The cyclic subrepresentation generated by this vector is the coherent representation, and since we have started from an irreducible representation this subrepresentation has to be the given one. \square

Note that the Theorem does *not* claim that all the coherent representations with specified r, α are positive. We strongly suspect that this is automatically the case, but were not able to prove it in full generality. However, some information about the coherent representations can be obtained with the methods we have introduced. From the proof of Proposition 3.2.2, we know that in every positive representation of (3.2.4) the relation $a_1^{*2}a_j^* = \mu^2 a_j^* a_1^{*2}$ holds. One readily verifies that this relation generates a Wick ideal. An even larger Wick ideal, which is also automatically annihilated is given in the following Proposition.

Proposition 3.2.4. *The Wick algebra determined by the μ CAR (3.2.4) has a Wick ideal, generated by the elements*

$$\begin{aligned} a_i a_1 a_1 - \mu^2 a_1 a_1 a_i & 1 < i \\ a_i a_i a_1 + \mu^{-1} a_i a_1 a_i - \mu a_i a_1 a_i - a_1 a_i a_i & 1 < i \\ a_j a_i a_1 + \mu^{-1} a_j a_1 a_i - \mu^2 a_i a_1 a_j - \mu a_1 a_i a_j & 1 < i < j \\ a_i a_j a_1 - \mu^2 a_j a_1 a_i - \mu a_1 a_j a_i - (-\mu^{-1} + \mu - \mu^3) a_i a_1 a_j - (1 - \mu^2) a_1 a_i a_j & 1 < i < j \end{aligned}$$

This ideal is annihilated in every positive representation. Consequently the coherent representation spaces are spanned by vectors of the form

$$a_{i_1}^* \cdots a_{i_n}^* \Phi_i, \quad (3.2.7)$$

with $i, i_\alpha > 1$, and

$$\Phi_i \in \{a_i^* \Omega, a_i^* a_1^* \Omega, a_1^* a_i^* \Omega, a_1^* a_i^* a_1^* \Omega\}.$$

Proof. The computations necessary to verify that these elements generate a Wick ideal are so tedious and so straightforward, that we performed them in MATHEMATICA [Mat]. The program is available by anonymous **ftp** (see the Introduction). That this ideal is annihilated in every positive representation follows immediately from Proposition 3.2.4 and Lemma 3.1.2. By using the generators of the ideal as substitution rules, we can transform any monomial in the a_i to a form in which all factors a_1 are collected on the left, apart from at most one intervening a_i with $i > 1$. Applying the adjoint of this statement to the coherent vector Ω , we obtain the second statement. \square

Further computations in MATHEMATICA showed that the scalar product in the coherent representation space is indeed strictly positive on some subspaces with low n in (3.2.7). However, we found no way to show this for general n . It would be particularly interesting to decide positivity for an infinite number of generators.

4. WICK ALGEBRA RELATIONS AS DIFFERENTIAL CALCULUS

Non-commutative differential calculus is a fast growing subject, and we cannot even begin to review it [Bae,Cu2,Man,Ros,Wo3]. There are some natural links with the structure of Wick algebras, however, and this section is devoted to describing these. In the first two subsections we will consider two *prima facie* different approaches to generalizing commutative differential calculus, one based on a generalization of the partial derivatives, the other on a generalization of the differential forms. In both cases Wick type relations appear naturally, and given these, the two approaches amount to the same thing. In the third subsection we will discuss two characterizations of the braid relations for T in terms of the resulting calculus, based on an idea of [WZ].

4.1. The algebra of differential operators.

Let us consider the tensor algebra $\mathcal{A} \equiv \mathcal{T}(\mathcal{H})$ as a non-commutative analogue of an algebra of “functions” generated by the coordinate functions x_i , $i \in I$. How can we generalize the commutative differential calculus to this algebra? The first approach to this problem is to build an analogue of the algebra *differential operators* on an abelian algebra. This is an associative algebra with identity, which is generated from the coordinates, considered as multiplication operators, together with the partial derivative operators ∂_i for $i \in I$. The structure of this algebra is determined by the commutation rule between coordinates and partial derivatives, and it is natural to assume

$$\partial_i x_j = \delta_{ij} \mathbb{1} + \sum_{k, \ell \in I} T_{ij}^{k\ell} x_\ell \partial_k \quad , \quad (4.1.1)$$

We notice that this is precisely our basic Wick algebra commutation rule with the substitution $i \mapsto x_i$, $i^\dagger \mapsto \partial_i$. We should point out that if we think of these relations as defining

a differential calculus, it is not clear why we should have a Hermitian Wick algebra. In the abelian case, we have the canonical commutation relations, which do define a Hermitian Wick algebra. Its basic positive representation is the Bargmann representation [Bar]. From this point of view it may be suggestive to assume hermiticity, but we will not do so for the moment.

The view of Wick algebras as differential calculi suggests some structures to investigate, which are useful for the study of Wick algebras independently of this interpretation. For $f \in \mathcal{T}(\mathcal{H})$, we can Wick order the product $\partial_i f$. This will produce a constant term and one containing exactly one partial derivative on the right:

$$\partial_i f = D_i(f) + \sum_{\ell} \Theta_i^{\ell}(f) \partial_{\ell} \quad . \quad (4.1.2)$$

Here D_i and Θ_i^{ℓ} are linear operators on $\mathcal{T}(\mathcal{H})$. One immediately gets the “twisted derivation” properties

$$\begin{aligned} D_i(fg) &= (D_i f)g + \sum_{\ell} \Theta_i^{\ell}(f) D_{\ell}(g) \\ \Theta_i^{\ell}(fg) &= \sum_k \Theta_i^k(f) \Theta_k^{\ell}(g) \quad , \end{aligned} \quad (4.1.3)$$

The second relation can be stated by saying that the *twist* $\Theta : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{M}_d(\mathcal{T}(\mathcal{H}))$ is a homomorphism into the algebra of $d \times d$ -matrices over $\mathcal{T}(\mathcal{H})$, where $d = |I|$ is the number of generators. We can also summarize the two relations (4.1.3) by saying that $\hat{\Theta} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{M}_{d+1}(\mathcal{T}(\mathcal{H}))$ is a homomorphism, where the matrix elements of $\hat{\Theta}$ are indexed by $I \cup \{0\}$, and are explicitly defined by

$$\hat{\Theta}_i^{\ell}(f) = \begin{cases} \Theta_i^{\ell}(f) & i, \ell \\ D_i(f) & \ell = 0, i \in I \\ 0 & i = 0 \end{cases} \quad . \quad (4.1.4)$$

In any case, it is clear that D and Θ can be computed by from (4.1.3), starting from the initial values

$$\begin{aligned} D_i \mathbb{I} &= 0 \quad , & D_i(x_j) &= \delta_{ij} \\ \Theta_i^{\ell}(\mathbb{I}) &= \delta_{i\ell} \quad , & \Theta_i^{\ell}(x_j) &= \sum_k T_{ij}^{\ell k} x_k \quad . \end{aligned} \quad (4.1.5)$$

To include the important case with algebraically dependent coordinates, we pass to a quotient $\mathcal{T}(\mathcal{H})/\mathcal{J}$ over a suitable ideal. We can still use the same commutation rules, provided the derivative and twist pass to the quotient, i.e. provided $D_i(\mathcal{J}) \subset \mathcal{J}$, and $\Theta_i^{\ell}(\mathcal{J}) \subset \mathcal{J}$. By Lemma 3.1.1(3), this is precisely the definition of a *Wick ideal*.

4.2. The algebra of differential forms.

A second independent route to a non-commutative differential calculus starts from the *differential forms* rather than the differential operators. The relevant structure here is that of a *graded differential algebra* [Con,Cu2]. The universal differential algebra over an algebra \mathcal{A} (here: $\mathcal{T}(\mathcal{H})$) is defined as the associative algebra $\Omega(\mathcal{A})$ generated by the elements f and df for $f \in \mathcal{A}$, where df depends linearly on f , $d\mathbb{1} = 0$, and $d(fg) = (df)g + fdg$. By applying this rule, every element of $\Omega(\mathcal{A})$ can be written in the form $\omega = f_0 df_1 \cdots df_p$. Such elements are defined to have degree p , for which we write $\deg \omega = p$, or $\omega \in \Omega^p(\mathcal{A})$. There is a unique extension of d to a linear operator on $\Omega(\mathcal{A})$, satisfying $d^2 = 0$, and the graded derivation property

$$d(\omega_1 \omega_2) = d(\omega_1) \omega_2 + (-1)^{\deg \omega_1} \omega_1 d(\omega_2) \quad , \quad (4.2.1)$$

applied whenever ω_1 is of homogeneous degree.

Note that while we can write every element as a linear combination of simple forms $f_0 df_1 \cdots df_p$, we cannot insist on a similar expansion in coordinate differentials dx_i , since in the universal algebra $\Omega(\mathcal{A})$ there is no commutation relation between x_i and dx_j . Let us postulate a relation of this kind in the general form

$$x_j dx_\ell = \sum_{ik} T_{ij}^{\ell k} (dx_i) x_k \quad . \quad (4.2.2)$$

We will denote by $\Omega_T = \Omega_T(\mathcal{T}(\mathcal{H}))$ the quotient of $\Omega(\mathcal{T}(\mathcal{H}))$ by this relation and its differential

$$dx_j dx_\ell + \sum_{ik} T_{ij}^{\ell k} dx_i dx_k = 0 \quad . \quad (4.2.3)$$

Thus Ω_T is the graded differential algebra with the universal property that for any homomorphism $\eta : \mathcal{T}(\mathcal{H}) \rightarrow \Omega'$ into another graded differential algebra Ω' , with the images $x'_i = \eta(x_i)$ satisfying (4.2.2), there is a unique graded homomorphism $\hat{\eta} : \Omega_T \rightarrow \Omega'$ extending η , and satisfying $d\hat{\eta} = \hat{\eta}d$.

By induction on the degree of a polynomial $f \in \mathcal{T}(\mathcal{H})$, we then find in Ω_T relations of the form

$$\begin{aligned} df &= \sum_i dx_i D_i(f) \\ f dx_\ell &= \sum_i (dx_i) \Theta_i^\ell(f) \quad , \end{aligned} \quad (4.2.4)$$

where D_i and Θ_i^ℓ are the linear operators on $\mathcal{T}(\mathcal{H})$, which are inductively defined by equations (4.1.5) and (4.1.3). It is clear that we can invert this construction, and define the algebra of differential forms Ω_T starting from the algebra of differential operators x_i, ∂_i , so these two approaches are essentially equivalent.

A consequence of equation (4.2.3) is that, for many of the Wick algebras considered in Section 1.2, the differential calculus collapses after the 1-forms, i.e. all forms of degree

≥ 2 vanish. A more precise description of the space of p -forms is given by the following Lemma.

Lemma 4.2.1. *Let Ω_0^p denote the space of p -forms with constant coefficients, i.e. the linear subspace of Ω_T generated by all expressions $dx_{i_1} \cdots dx_{i_p}$. Then Ω_0^p is canonically isomorphic to*

$$\bigcap_{r=1}^{p-2} \mathcal{H}^{\otimes(r-1)} \otimes (\ker(\mathbb{I} + T^*)) \otimes \mathcal{H}^{\otimes(p-r-1)} \subset \mathcal{H}^{\otimes p} \quad . \quad (4.2.5)$$

Proof. Ω_0^p is the quotient of the space of linear combinations $\sum \Phi(i_1, \dots, i_p) dx_{i_1} \cdots dx_{i_p}$ by the subspace generated by the coefficients Φ of the form

$$\sum_{k\ell} \langle i_r i_{r+1} | \mathbb{I} + T | k\ell \rangle \Psi(i_1, \dots, i_r, k, \ell, i_{r+2}, \dots, i_p) = ((\mathbb{I} + T_r)\Psi)(i_1, \dots, i_p) \quad .$$

This subspace is the span of the ranges of the operators $(\mathbb{I} + T_r)$. Since in a Hilbert space we can identify the quotient by a subspace with its orthogonal complement, we get

$$\Omega_0^p \cong \left(\bigcup_r (\mathbb{I} + T_r) \mathcal{H}^{\otimes p} \right)^\perp = \bigcap_r ((\mathbb{I} + T_r) \mathcal{H}^{\otimes p})^\perp \quad .$$

□

Let us consider the space Ω_0^p in the examples. In **Example 1.2.1** the kernel of $\mathbb{I} + T$ is zero, so $\Omega_0^p = \{0\}$ for all $p \geq 2$, and the same conclusion holds for any Wick algebra with small T (**Example 1.2.9**). In **Example 1.2.2** we may choose $q = -d^{-1}$, and get

$$dx_i dx_j = \delta_{ij} \frac{1}{d} \sum_k dx_k dx_k \equiv \delta_{ij} \omega \quad . \quad (4.2.6)$$

On the other hand, one easily verifies that $\omega dx_i = d^{-1} dx_i \omega = d^{-2} \omega dx_i$. Hence $\Omega_0^2 = \mathbb{C}\omega$, and $\Omega_0^p = \{0\}$ for $p \geq 3$.

For the twisted canonical commutation relations, i.e. the Bosonic case of **Example 1.2.3**, the kernel of $\mathbb{I} + T$ is generated by the vectors $|ij\rangle - \mu|ji\rangle$, with $i < j$. In particular, any form in which some dx_i appears twice vanishes. As in the commutative case, which corresponds to $\mu = 0$, we get $\dim \Omega_0^p = \binom{d}{p}$, where d is the number of generators. In the Fermionic case (twisted anti-commutation relations) we get the same combinatorial problem as in the determination of the dimension of the Bosonic Fock space over a d -dimensional one-particle space. Thus

$$\dim \Omega_0^p = (-1)^p \binom{-d}{p} = \binom{d+p-1}{p} \quad . \quad (4.2.7)$$

Since the Clifford algebras (**Example 1.2.6**) are the special case with $\mu = 1$, this formula also holds in that case. These examples follow a general pattern: the sign in the graded derivation property (4.2.1) forces the differentials of commuting variables to anti-commute, and conversely. Moreover, this duality persists under μ -deformation.

One consequence of (4.2.7) is that $\Omega_0^p \neq \{0\}$ for all p . A more trivial example with this property is **Example 1.2.7**, where $\dim \Omega_0^p = d^p$ even grows exponentially. However, if we replace $\ker(\mathbb{I} + T^*)$ by a generic subspace of $\mathcal{H} \otimes \mathcal{H}$ the intersection in (4.2.5) vanishes for large p . The geometric property of this subspace making $\Omega_0^p \neq \{0\}$ for all p has been investigated recently as a property of finite range interactions of quantum spin chains, where it leads to a new class of exactly solvable ground state problems [We2,FNW]. A paradigm of a subspace with this property [AKLT] is the sum of the spin-0 and spin-1 subspaces of a spin-1 chain. This corresponds to a Wick algebra on three generators with relations

$$i^\dagger j = \delta_{ij} \left(1 - \frac{\lambda - 2}{2} \sum_k k k^\dagger \right) + \frac{\lambda}{2} j i^\dagger - \frac{\lambda}{3} i j^\dagger \quad . \quad (4.2.8)$$

In that case we have $\dim \Omega_0^p = 4$ for all $p \geq 2$.

4.3. Differential calculus with braid relations.

Wess and Zumino [WZ] have proposed to “complete the algebra” generated by the three kinds of objects x_i , ∂_i , and dx_i by introducing commutation relations between ∂_i and dx_i . This would indeed be natural if we want to upgrade the partial derivatives ∂_i to “*covariant derivatives*” acting also on differential forms. They make the ansatz

$$\partial_i dx_j = \sum_{k,\ell} S_{ij}^{k\ell} dx_\ell \partial_k \quad , \quad (4.3.1)$$

where the $S_{ij}^{k\ell}$ are complex coefficients. If we apply relations (4.1.1), (4.2.2), and (4.3.1) to $\partial_i x_j dx_k$ we have two different routes for applying the rules which agree if and only if

$$\sum_\ell dx_\ell \langle i\ell | ST - \mathbb{I} | jk \rangle = 0 \quad (4.3.2a)$$

$$\sum_{\ell nm} dx_\ell x_m \partial_m \langle i\ell n | T_2 T_1 S_2 - S_1 T_2 T_1 | jkm \rangle = 0 \quad , \quad (4.3.2b)$$

where $S : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ has the matrix elements

$$\langle i\ell | S | jk \rangle = S_{ij}^{k\ell} \quad . \quad (4.3.3)$$

Now if we want the differentials dx_ℓ , as well as the combinations $dx_\ell x_m \partial_m$ to be linearly independent in the resulting algebra, we have to take $ST = \mathbb{I}$ from (4.3.2a), and taking the number of generators finite, we also get $TS = \mathbb{I}$. We emphasize that the existence of this inverse has nothing to do with the possibility of solving the Wick relations for the

right hand side, thereby defining an “anti-Wick ordering”. This transformation would be connected with the existence \tilde{T}^{-1} , with \tilde{T} from (1.1.5) rather than $S = T^{-1}$ with T from (2.1.2). It is easy to find examples, where \tilde{T} is invertible, while T is not, and conversely.

Writing the expression in (4.3.2b) as $S_1(T_1T_2T_1 - T_2T_1T_2)S_2 = 0$, we get the braid condition $T_1T_2T_1 = T_2T_1T_2$ for T . The verification that the braid condition indeed suffices to define an algebra from relations (4.1.1), (4.2.2), and (4.3.1) can be done using the Diamond Lemma [Ber], in the same way as in the proof of Theorem 4.3.2 below.

If we read ∂_i as x_i^\dagger , it is natural to try to extend the algebra still further, by including the adjoint differentials dx_i^\dagger . We would thus arrive at a structure containing both operations, involution as well as differential. It turns out that the development in this section has put us on the wrong track for doing this. Roughly, the reason is the following: since we have a relation transforming $x^\dagger dx$ to $dx x^\dagger$, with coefficients S , its adjoint should transform $x dx^\dagger$ to $dx^\dagger x$ with coefficients S^* . Taking the differential of either relation we get a rule transforming $dx^\dagger dx$ to $dx dx^\dagger$, and the two possibilities match only when $S = S^*$. But when we apply these rules to the differential $d(x_i^* x_j)$, and compare with the basic Wick relation, we find that we should have $S = T$, rather than $S = T^{-1}$. Nevertheless, the braid condition is sufficient to define a structure allowing both involution and differentials:

Definition 4.3.1. *Let I be finite, and let $T_{ij}^{k\ell}$, $i, j, k, \ell \in I$, be the coefficients of a Hermitian Wick algebra $\mathcal{W}(T)$. Then a **Wick differential *-algebra** over T is an associative algebra $\Omega\mathcal{W}$ with the following properties:*

- (1) $\Omega\mathcal{W}$ is generated by elements “ f ”, and “ df ”, for $f \in \mathcal{W}(T)$, where “ df ” depends linearly on f .
- (2) $d\mathbb{I} = 0$, and $d(fg) = (df)g + fdg$.
- (3) $\Omega\mathcal{W}$ has an involution \dagger extending that of $\mathcal{W}(T)$ such that $d(f^\dagger) = (df)^\dagger$.
- (4) There are complex coefficients $S_{ij}^{k\ell}, R_{ij}^{k\ell}$ such that

$$x_i dx_j = \sum_{k\ell} R_{ij}^{k\ell} dx_\ell x_k \quad (4.3.4a)$$

$$x_i^\dagger dx_j = \sum_{k\ell} S_{ij}^{k\ell} dx_\ell x_k^\dagger \quad (4.3.4b)$$

- (5) Monomials of the form

$$dx_{i_1} \cdots dx_{i_m} x_{j_1} \cdots x_{j_n} x_{k_p}^\dagger \cdots x_{k_1}^\dagger dx_{\ell_q}^\dagger \cdots dx_{\ell_1}^\dagger, \quad (4.3.5)$$

with $i_\alpha, j_\alpha, k_\alpha, \ell_\alpha \in I$, and $m, n, p, q \geq 0$, span $\Omega\mathcal{W}$.

- (6) The monomials (5) are linearly independent modulo the relations

$$dx_i dx_j = - \sum_{k\ell} R_{ij}^{k\ell} dx_\ell dx_k, \quad i, j \in I, \quad ,$$

and their adjoints.

Theorem 4.3.2. *Let $T_{ij}^{k\ell}$ be as in Definition 4.3.1, $\mathcal{H} = \mathbb{C}^I$, and define the operators $T, R, S : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ by*

$$T_{ij}^{k\ell} = \langle i\ell | T | jk \rangle \quad , \quad S_{ij}^{k\ell} = \langle i\ell | S | jk \rangle \quad , \quad \text{and} \quad R_{ij}^{k\ell} = \langle \ell k | R | ij \rangle \quad .$$

Then a Wick differential $$ -algebra over T exists if and only if T is invertible, and satisfies the braid relation $T_1 T_2 T_1 = T_2 T_1 T_2$. In this case, the S and R must be chosen to be $S = T$, and $R = T^{-1}$, and $\Omega\mathcal{W} \equiv \Omega\mathcal{W}(T)$ is uniquely determined by T .*

Proof. By taking adjoints and differentials of the relations (4.3.4) we get the following system of equations:

$$x_i^\dagger x_j = \delta_{ij} \mathbb{1} + \sum_{k\ell} \langle i\ell | T | jk \rangle x_\ell x_k^\dagger \quad (4.3.6a)$$

$$x_i dx_j = \sum_{k\ell} \langle \ell k | R | ij \rangle dx_\ell x_k \quad (4.3.6b)$$

$$x_i^\dagger dx_j = \sum_{k\ell} \langle i\ell | S | jk \rangle dx_\ell x_k^\dagger \quad (4.3.6c)$$

$$dx_i^\dagger x_j = \sum_{k\ell} \langle i\ell | S^* | jk \rangle x_\ell dx_k^\dagger \quad (4.3.6d)$$

$$dx_i^\dagger x_j^\dagger = \sum_{k\ell} \langle ji | R^* | k\ell \rangle x_\ell^\dagger dx_k^\dagger \quad (4.3.6e)$$

$$dx_i^\dagger dx_j = - \sum_{k\ell} \langle i\ell | S^* | jk \rangle dx_\ell dx_k^\dagger \quad (4.3.6f)$$

$$dx_i dx_j = - \sum_{k\ell} \langle \ell k | R | ij \rangle dx_\ell dx_k \quad (4.3.6g)$$

$$dx_i^\dagger dx_j^\dagger = - \sum_{k\ell} \langle ij | R^* | k\ell \rangle dx_\ell^\dagger dx_k^\dagger \quad (4.3.6h)$$

We can obtain (4.3.6f) as the differential of either (4.3.6c) or (4.3.6d). These two have to agree, so that we must have $S = S^*$. Taking the differential of (4.3.6a), and substituting (4.3.6d) and (4.3.6c), we find that $S = T$. Thus the system (4.3.6) becomes closed under both adjoints and differentials.

The crucial part of the proof is to apply the Diamond Lemma [Ber] to this system, omitting the last two. The “Wick ordered form” in which no further substitutions are possible is the form given in (4.3.5). It is obvious that the substitution process terminates. We can apply any of the rules, whenever we find two factors $x, x^\dagger, dx, dx^\dagger$ in the “wrong” order. If we find two occurrences of this in the same monomial, in two non-overlapping pairs of factors, it is clear that either of the two choices leads to the same result. The only situations we have to check therefore, are those of three adjacent factors, which are in the opposite order. Since we have only three types of factors, there are only four types of such triples. Of these two are adjoints of the other pair, which leaves us with checking the consistency of the two possible reduction paths for the two types of products $x_i^\dagger x_j dx_k$, and $dx_i^\dagger x_j dx_k$. Now the first amounts to two conditions, one arising from the first term in (4.3.6a), and one from the second. These are $RT = \mathbb{1}$, and the braid relation, respectively. The condition for the second type of triple follows from these.

Hence the algebra defined by relations (4.3.6.a-f) has the required properties. It remains to be checked that (4.3.6.g,h) are compatible with this, i.e. that it makes no difference

whether we apply these relations before of after Wick ordering. For $x dx dx \rightarrow dx dx x$ this follows from the braid relation for R . Since the coefficients for the commutation of coordinates and differentials are the same as for the commutation of two differentials, the compatibility of (4.3.6g) with the transformation $x^\dagger dx dx \rightarrow dx dx x^\dagger$ follows by virtually the same computation as for $x^\dagger x dx \rightarrow dx x x^\dagger$. For the same reason we need not check compatibility for $dx^\dagger dx dx$, and $dx dx dx$, and the compatibility for relation (4.3.6h) follows by taking adjoints. \square

Since $\Omega\mathcal{W}$ is an involutive algebra we may once again ask for *positive* representations. Rather than aiming at the development of a general theory, we state some basic features of the representation theory of $\Omega\mathcal{W}(T)$. Note that, in spite of the difference between (4.2.2) and (4.3.4a) with $R = T^{-1}$, Lemma 4.2.1 applies without change, because $(\mathbb{1} + T)$ and $(\mathbb{1} + T^{-1})$ have the same kernel.

Proposition 4.3.3. *Let $\Omega\mathcal{W}(T)$ be a differential Wick algebra over T .*

- (1) *Then if $\Omega_0^p = \{0\}$ for some $p \geq 2$, we have $\pi(dx_i) = \pi(dx_i^\dagger) = 0$ for all i .*
- (2) *There is no universal bounded representation of $\Omega\mathcal{W}(T)$.*

Proof. (1) Let π be a positive representation of $\Omega\mathcal{W}(T)$, and let q be the largest integer such that there is a q -form $\omega \in \Omega_0^p$ with $\pi(\omega) \neq 0$. Clearly, $q < p < \infty$. On the other hand, let $\omega \in \Omega_0^{q'}$ with $q/2 < q' \leq q$. Then $\pi((\omega^\dagger \omega)^2) = 0$, because after Wick ordering this expression contains a form of degree $2q' > q$. Hence in $\pi(1 + \lambda \omega^\dagger \omega)^2$ the term quadratic in λ vanishes, and, by positivity, the linear term has to vanish, as well. Hence $\pi(\omega) = 0$, contradicting the minimality of q , unless $q = 0$.

(2) Let π be a bounded representation. Then we can obtain a new one, $\tilde{\pi}$ by setting $\tilde{\pi}(x_i) = \pi(x_i)$, and $\tilde{\pi}(dx_i) = \lambda \pi(dx_i)$, with $\lambda \in \mathbb{C}$ arbitrary. Hence there cannot be a universal bound on $\|dx_i\|$, which holds in every bounded representation. \square

Note that the first item excludes the twisted canonical commutation relations (Example 1.2.3), by the discussion at the end of Section 4.2. On the other hand, the differential Wick algebra of the twisted canonical anti-commutation relations (μ CAR) seem to have non-trivial representations. This is certainly the case for Clifford algebras (Example 1.2.6), as the following example shows: consider the Fock representation of the (untwisted) canonical anti-commutation relations, whose scalar product is clearly positive. Denote by Z the operator defined by $Z\Omega = \Omega$, and $Zx_i = -x_i Z$. Then we can define $dx_i = \xi_i Z$, where $\xi_i \in \mathbb{C}$ is an arbitrary constant, and it is immediately verified that the relations (4.3.4) and (4.3.6) hold.

5. GAUGE AUTOMORPHISMS AND THEIR KMS STATES

From the universal property of $\mathcal{W}(T)$ we see that any Wick algebra has a † -automorphism group $\alpha_t : \mathcal{W}(T) \rightarrow \mathcal{W}(T)$ defined by

$$\alpha_t(k) = e^{it}k \quad , \quad \text{for } t \in \mathbb{R}, \text{ and } k \in I, \quad (5.1)$$

which we call the *gauge group* of $\mathcal{W}(T)$. That α_t is a † -automorphism is equivalent to $\alpha_t(k^\dagger) = e^{-it}k^\dagger$. The pattern for defining these automorphisms Wick algebras sometimes applies also to other linear transformations of the generators, depending on T . The only thing one needs to check is the invariance of the elements $i^\dagger \otimes j - \delta_{ij}\mathbf{1} - \sum T_{ij}^{k\ell} \ell \otimes k^\dagger$ in the tensor algebra $\mathcal{T}(\mathcal{H}, \mathcal{H}^\dagger)$. The definition of the corresponding automorphism is then clear from the universal property. What makes the gauge transformations special is that this scheme works for *all* Wick algebras. The gauge automorphisms also extend to the universal bounded representation $\mathbf{W}(T)$, if the latter exists, and we will then denote the corresponding C^* -automorphism group by α_t , as well.

The gauge group defines on $\mathcal{W}(T)$ a natural \mathbb{Z} -grading by

$$\deg(X) = n \quad \Longleftrightarrow \quad \alpha_t(X) = e^{int}X \quad . \quad (5.2)$$

When we use the notation $\deg(X)$ it is implied that the element X is indeed of homogeneous degree. On a monomial in the generators the degree is computed simply by counting the number of unstarred generators k and subtracting the number of starred ones k^\dagger in the monomial. The degree zero, or gauge invariant part of $\mathcal{W}(T)$ (resp. $\mathbf{W}(T)$) is denoted by $\mathcal{W}(T)^\alpha$ (resp. $\mathbf{W}(T)^\alpha$). There is a faithful conditional expectation from $\mathbf{W}(T)$ onto $\mathbf{W}(T)^\alpha$, defined by

$$X \longmapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} dt \, \alpha_t(X) \quad . \quad (5.3)$$

Definition 5.1. *Let $\mathcal{W}(T)$ be a Wick algebra, and $\lambda \in \mathbb{R}$. Then a **gauge KMS-functional** at fugacity λ is a linear functional τ_λ on $\mathcal{W}(T)$ such that $\tau_\lambda(\mathbf{1}) = 1$,*

$$\tau_\lambda(kX) = \lambda \tau_\lambda(Xk) \quad , \quad \text{and} \quad \tau_\lambda(Xk^\dagger) = \lambda \tau_\lambda(k^\dagger X) \quad .$$

If this functional extends to $\mathbf{W}(T)$, it will be denoted by the same letter.

On the C^* -algebra $\mathbf{W}(T)$, states with this property are indeed precisely the KMS-states for the gauge group. To see this, note that the polynomials in the generators form a dense subalgebra of $\mathbf{W}(T)$ on which the gauge group is even analytic with $\alpha_{i\beta}(X) = \exp(-\beta \deg(X))X$. Now one of the equivalent versions of the KMS-condition at inverse temperature β (e.g. 5.3.1 in [BR]) requires the existence of such a dense subalgebra of analytic elements, and the condition $\omega(A\alpha_{i\beta}(B)) = \omega(BA)$, for A, B in that subalgebra. Hence the KMS-condition for is equivalent to

$$\omega(XA) = e^{-\beta \deg(X)} \omega(AX) \quad , \quad (5.4)$$

for $A \in \mathbf{W}(T)$, and X a homogeneous polynomial. By induction on the degree of X this is easily seen to be equivalent to the definition of τ_λ , with $\lambda = e^{-\beta}$. (The expression “fugacity” for the parameter λ is borrowed from statistical mechanics: in the language of physics the gauge group is generated by the “number operator”, $-\beta$ is called the “chemical potential”, and “fugacity” for λ is standard terminology. The case $\lambda = 0$ defines the Fock state, which is the unique ground state [BR] for α_t , formally corresponding to $\beta = \infty$, or temperature $1/\beta = 0$). Negative λ are excluded by the following Lemma.

The most striking consequence of the Wick algebra structure for the gauge-KMS states is that we obtain an explicit prescription for computing τ_λ :

Lemma 5.2. *Let $\mathcal{W}(T)$ be a Wick algebra, and $\lambda \in \mathbb{R}$, and let τ_λ be a gauge KMS-functional at fugacity λ . Then*

- (1) *For $\lambda \neq 1$, $\deg(X) \neq 0$ implies $\tau_\lambda(X) = 0$.*
- (2) *If $\tau_\lambda(X^\dagger X) \geq 0$ for all $X \in \mathcal{W}(T)$, we must have $\lambda \geq 0$.*
- (3) *Suppose that $\|\tilde{T}\| \leq 1$, and $|\lambda| < \|\tilde{T}\|^{-1}$ where $\tilde{T} : \mathcal{H}^\dagger \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}^\dagger$ is the operator introduced in equation (1.1.5). Then τ_λ exists and is uniquely determined.*
- (4) *Suppose that $\|\tilde{T}\| \leq 1$. Then, for any $X \in \mathcal{W}(T)$, $\lambda \mapsto \tau_\lambda(X)$ is an analytic function for $|\lambda| < \|\tilde{T}\|^{-1}$.*

Proof. (1) $\tau_\lambda(X) = \tau_\lambda(X\mathbb{1}) = \lambda^{\deg(X)}\tau_\lambda(\mathbb{1}X)$, hence $\lambda^{\deg(X)} = 1$, whenever $\tau_\lambda(X) \neq 0$.

(2) When τ_λ is positive, $\tau_\lambda(kk^\dagger) = \lambda\tau_\lambda(k^\dagger k)$ implies that either $\lambda \geq 0$ or $\tau_\lambda(k^\dagger k) = 0$ for all $k \in I$. The latter condition would imply $\tau_\lambda(XkY) = \lambda^{-\deg(Y)}\tau_\lambda(YXk) = 0$, so all k would be annihilated by the GNS-representation with respect to τ_λ , which contradicts the Wick relations (1.1.1).

(3) The prescription for evaluating τ_λ on all Wick ordered monomials of the form $X = i_1 \cdots i_n j_m^\dagger \cdots j_1^\dagger$ is the following: we exchange the two groups of generators, to get $\tau_\lambda(X) = \lambda^n \tau_\lambda(\tilde{X})$, where $\tilde{X} = j_m^\dagger \cdots j_1^\dagger i_1 \cdots i_n$. Note that $\lambda^n = \lambda^m$, or $\tau_\lambda(X) = 0$ by (1). Now we Wick order \tilde{X} . This produces a linear combination of Wick ordered monomials with degrees $n' \leq n, m' \leq m$. We collect all terms of the same degree as X . This leading term is computed by using the Wick relations without constant term, and is described by an operator

$$\tilde{T}^{(n,m)} : (\mathcal{H}^\dagger)^{\otimes m} \otimes \mathcal{H}^{\otimes n} \longrightarrow \mathcal{H}^{\otimes n} \otimes (\mathcal{H}^\dagger)^{\otimes m},$$

inductively defined from $\tilde{T}^{(1,1)} = \tilde{T}$ from equation (1.1.5). $\tilde{T}^{(n,m)}$ is the product of nm copies of \tilde{T} , acting in appropriate tensor factors. Hence $\|\tilde{T}^{(n,m)}\| \leq \|\tilde{T}\|^{nm}$. Now

$$\tau_\lambda(X - \lambda^n \tilde{T}^{n,m} \tilde{X}) = \tau_\lambda(\text{terms of lower degree})$$

is an inductive formula for τ_λ provided the operator $X \mapsto (X - \lambda^n \tilde{T}^{n,m} \tilde{X})$ is invertible on $\mathcal{H}^{\otimes n} \otimes (\mathcal{H}^\dagger)^{\otimes m}$. Clearly, it is sufficient for this that $|\lambda|^n \|\tilde{T}^{(n,m)}\| < (|\lambda| \|\tilde{T}\|)^n \|\tilde{T}\|^m < 1$, which is in turn implied by the assumption made in (3).

(4) is immediate from (3) and the analyticity of the inverse operator used in (3). \square

For special choices of T the uniqueness result can be improved. What we have used in the proof is only the invertibility of the operator $X \mapsto (X - \lambda^n \tilde{T}^{n,m} \tilde{X})$. For the q -relations (Example 1.2.1) this is simply $(1 - \lambda^n q^{nm})\mathbb{I}$. Hence in this case uniqueness holds whenever $\lambda \neq q^{-m}$ for all m . This includes the value $\lambda = 1$, for which τ_λ is a trace, and almost all $\lambda > 1$ (for which the “temperature” $\beta = -\ln \lambda$ is negative). On the other hand, we know that the algebra $\mathbf{W}(T)$ in this case has no tracial states. Hence the functional τ_1 cannot be positive. The following Theorem collects the basic results on the positivity of τ_λ .

Theorem 5.3. *Let $\mathcal{W}(T)$ be a Wick algebra with $d = |I| < \infty$ generators, and suppose that the Fock state $\omega_0 = \tau_0$ is positive. Let P_n be the operators on $\mathcal{H}^{\otimes n}$ defined in (2.1.4). Then the following conditions are equivalent:*

- (1) τ_λ is positive, and normal with respect to the Fock representation.
- (2) $\sum_{n=0}^{\infty} \lambda^n \text{rank } P_n < \infty$.

In particular, τ_λ is positive for $0 \leq \lambda \leq 1/d$, and if $P_n > 0$ for all n , $\tau_{1/d}$ is not normal with respect to the Fock representation.

Proof. In the Fock representation the gauge group is implemented by the number operator N , defined by $(N - n)i_1 \cdots i_n \mathbb{I} = 0$. Then if the operator λ^N is trace class, it defines the density matrix of a KMS state τ_λ for α_t . The converse holds, because the Fock state is pure, and hence the Fock representation is irreducible. Hence (1) is equivalent to

$$\text{tr } \lambda^N = \sum_{n=0}^{\infty} \lambda^n \text{rank } P_n < \infty \quad .$$

Since $\text{rank } P_n \leq d^n$, this criterion holds when $\lambda d < 1$. Moreover, when $P_n > 0$, $\text{rank } P_n = d^n$, and the criterion fails for $\lambda d = 1$. \square

With the help of this Theorem we can now discuss the existence of gauge KMS-states for the examples in Section 1.2. For the generalized q -relations (Example 1.2.5, (1.2.11)) we have $\|\tilde{T}\| = \max_{ij} |q_{ij}|$. Hence for constant $q_{ij} \equiv q$, and $|q| < 1$ we have $\|\tilde{T}\| < 1$, and we have unique KMS states for $0 \leq \lambda \leq 1/d$. For $q = 0$ it is easy to see (Example 5.3.27 in [BR]) that there cannot be a state τ_λ with $\lambda > 1/d$. More generally, we know for any sufficiently small T , i.e. for T satisfying the hypothesis of Theorem 2.4.4, the exact range of the parameter λ for which $\tau_\lambda \geq 0$. This follows readily from the isomorphism $\mathbf{W}(T) \cong \mathbf{W}(0)$ established in that Theorem, and the observation that the isomorphism intertwines the respective gauge groups. The growth $\text{rank } P_n = d^n$ is typical for all these examples. It is also responsible for the anomalous thermodynamical behaviour of the “free quon gas” [We1].

For the twisted canonical (anti-) commutation relations the dimension of the n -particle Fock space grows exactly as for the untwisted counterparts. This agreement of the “Poincaré

series” [WZ] with the undeformed case, is, in fact, one of the motivations for studying just this type of deformation. Thus we have

$$\mathrm{tr}(\lambda^N) = \begin{cases} (1 - \lambda)^{-d} & \text{for (1.2.3)} \\ (1 + \lambda)^d & \text{for (1.2.4).} \end{cases}$$

Hence the range of λ is $[0, 1]$ in the Bosonic case, and $[0, \infty]$ in the Fermionic case ($\lambda = \infty$ defines a “ceiling state” [BR]).

The state τ_1 in the Bosonic case, which is positive by continuity of τ_λ in λ , is a trace. Indeed, the algebra $\mathbf{W}(T)$ in this case has a tracial state. In the corresponding GNS-representation all generators are zero, except the first, which is unitary up to a factor.

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REFERENCES

- [AKLT] I. Affleck, T. Kennedy, E.H. Lieb, and H. Tasaki, *Valence bond ground states in isotropic quantum antiferromagnets*, Commun.Math.Phys. **115** (1988), 477–528.
- [Bae] J.C. Baez, *Differential calculi on quantum vector spaces with Hecke-Type relations*, Lett.Math. Phys. **23** (1991), 133–141; ———, *R-commutative geometry and quantization of Poisson algebras*, Adv.Math. **95** (1992), 61–91.
- [Bar] V. Bargmann, *Group representations on Hilbert spaces of analytic functions*, pp. 27–63, Analytic Methods in Mathematical Physics (R.P. Gilbert, ed.), Gordon Breach, New York, 1970, pp. 27–63; ———, *On a Hilbert space of analytic functions and an associated integral transform*, Comm.Pure Appl.Math. **14** (1961), 187–214.
- [Ber] G. Bergmann, *The diamond lemma for ring theory*, Adv.Math. **29** (1978), 178–218.
- [BS1] M. Bożejko and R. Speicher, *An example of a generalized Brownian motion*, Commun. Math.Phys. **137** (1991), 519–531; partII:L. Accardi (ed.), Quantum Probability & related topics VII, World Scientific, Singapore, 1992, pp. 67–77.
- [BS2] ———, *Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces*, Preprint Heidelberg, Summer 1993.
- [Bla] B. Blackadar, *Shape theory for C^* -algebras*, Math.Scand. **56** (1985), 249–275.
- [BEIGJ] O. Bratteli, G.A. Elliott, F.M. Goodman, and P.E.T. Jørgensen, *On Lie algebras of operators*, J.Funct.Anal. **85** (1989), 341–359.
- [BEvGJ] O. Bratteli, D.E. Evans, F.M. Goodman, and P.E.T. Jørgensen, *A dichotomy for derivations on \mathcal{O}_n* , Publ. RIMS, Kyoto Univ. **22** (1986), 103–117.
- [BR] O. Bratteli and D.W. Robinson, *Operator algebras and quantum statistical mechanics*, volume II, Springer Verlag, Berlin, Heidelberg, New York, 1981.
- [Con] A. Connes, *Non-commutative differential geometry*, Publ.Math.IHES **62** (1985), 257–360.

- [Cu1] J. Cuntz, *Simple C^* -algebras generated by isometries*, Commun.Math.Phys. **57** (1977), 173–185.
- [Cu2] ———, *Quantized differential forms in non-commutative topology and geometry*, AMS series Contemporary Mathematics, vol. 145, 1993, pp. 65–78; ———, *Representations of quantized differential forms in non-commutative geometry*, Mathematical Physics X (K. Schmüdgen, ed.), Springer-Verlag, Berlin, 1992, pp. 237–251.
- [Das] C. Daskaloyannis, *Generalized deformed oscillator and nonlinear algebras*, J. Phys. A **24** (1991), L789–L794.
- [Don] W.F. Donoghue, Jr., *Monotone matrix functions and analytic continuation*, Grundlehren, vol 207, Springer-Verlag, Berlin, 1974.
- [DN] K. Dykema and A. Nica, *On the Fock representation of the q -commutation relations*, J. reine angew. Math. **440** (1993), 201–212.
- [FRT] L. Faddeev, N. Reshetikhin, and L. Takhtajan, *Quantum groups, Braid groups, knot theory, and statistical mechanics* (C.N. Yang, ed.), World Scientific, Singapore, 1989, pp. 97–100.
- [FNW] M. Fannes, B. Nachtergaele, and R.F. Werner, *Finitely correlated states of quantum spin chains*, Commun.Math.Phys. **144** (1992), 443–490; ———, *Finitely correlated pure states*, J.Funct. Anal., archived in mp_arc@math.utexas.edu, #92-132 (to appear).
- [Fiv] D.I. Fivel, *Interpolation between Fermi and Bose statistics using generalized commutators*, Phys.Rev.Lett. **65** (1990), 3361–3364; Erratum, Phys.Rev.Lett. **69** (1992), 2020.
- [Fol] G.B. Folland, *Harmonic Analysis in Phase Space*, Princeton Univ. Press, Princeton, 1989.
- [GW] R.W. Goodman and N.R. Wallach, *Projective unitary positive-energy representations of $Diff(S^1)$* , J.Funct.Anal. **63** (1985), 299–321.
- [Gre] O.W. Greenberg, *Particles with small violations of Fermi or Bose statistics*, Phys.Rev.D. **43** (1991), 4111–4120.
- [Jo1] V.F.R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann.Math. **126** (1987), 335–388.
- [Jo2] ———, *Index for subfactors*, Invent.Math. **72** (1983), 1–25.
- [Jor] P.E.T. Jørgensen, *Operators and Representation Theory*, North Holland (Elsevier), Amsterdam, 1988.
- [JW] P.E.T. Jørgensen and R.F. Werner, *Coherent states of the q -canonical commutation relations*, Preprint Osnabrück 1993, archived at funct-an@babbage.sissa.it, #9303002.
- [JSW1] P.E.T. Jørgensen, L.M. Schmitt, and R.F. Werner, *q -canonical commutation relations and stability of the Cuntz algebra*, Pacific.J.Math., archived at mp_arc@math.utexas.edu, #93-52 (to appear).
- [JSW2] ———, *q -relations and stability of C^* -isomorphism classes*, Algebraic methods in operator theory (R.E. Curto and P.E.T. Jørgensen, eds.), Birkhäuser Verlag, 1993.
- [KR] V.G. Kac and A.K. Raina, *Highest Weight Representations of infinite-dimensional Lie Algebras*, World Scientific, Singapore, 1987.
- [KL] S. Klimek and A. Lesniewski, *A two-parameter deformation of the unit disc*, J.Funct.Anal. **115** (1993), 1–23.
- [KW] P. Kruszyński and S.L. Woronowicz, *A non-commutative Gelfand-Naimark theorem*, J.Operator Th. **8** (1982), 361–389.
- [Man] Yu. I. Manin, *Topics in Non-Commutative Geometry*, Princeton Univ. Press, Princeton, 1991.
- [Mat] Wolfram Research, Inc., *Mathematica 2.2*, Wolfram Research, Inc., Champaign, Illinois, 1992.
- [Mei] A. Meister, *The C^* -algebra obtained by completing the quantum group $Fun_q(SU(2))$* , J.Math. Phys. **33** (1992), 4177–4189.
- [NF] B. Sz-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland (Elsevier), Amsterdam, 1970.
- [NN] G. Nagy and A. Nica, *On the ‘quantum disk’ and a ‘non-commutative circle’*, Preprint, Berkeley 1992.
- [Phi] N.C. Phillips, *Inverse limits of C^* -algebras and applications*, Operator algebras and application. Vol. 1 (D.E. Evans and M. Takesaki, eds.), Lond.Math.Soc.Lect.Note Ser., vol. 135, Cambridge University P., Cambridge, 1988.
- [Pus] W. Pusz, *Twisted canonical anticommutation relations*, Rep.Math.Phys. **27** (1989), 349–360.
- [PW] W. Pusz and S.L. Woronowicz, *Twisted second quantization*, Rep.Math.Phys. **27** (1989), 231–

- 257.
- [RR] H. Radjavi and P. Rosenthal, *Invariant subspaces*, Ergebnisse, vol. 77, Springer-Verlag, Berlin, New York, 1973.
 - [RS] M. Reed and B. Simon, *Methods in Modern Mathematical Physics, I: Functional Analysis*, Academic Press, New York, 1980.
 - [Reh] K.-H. Rehren, *Charges in quantum field theory*, Mathematical Physics X (K. Schmüdgen, ed.), Springer-Verlag, Berlin, 1992, pp. 388–392.
 - [Ros] A.L. Rosenberg, *The spectrum of the algebra of skew-differential operators and the irreducible representations of the quantum Heisenberg algebra*, Commun.Math.Phys. **142** (1991), 567–588.
 - [Rud] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.
 - [SW] L.M. Schmitt and R.F. Werner, *Non-commutative functional calculus*, in preparation.
 - [Sha] D. Shale, *Linear symmetries of free boson fields*, Trans.Amer.Math.Soc. **103** (1962), 149–167.
 - [Smü] K. Schmüdgen, *Über LMC*-Algebren*, Math.Nachr. **68** (1975), 167–182.
 - [Slo] W. Slowikowski, *Ultracoherence in Bose algebras*, Adv.Appl.Math. **9** (1988), 377–427.
 - [Sp2] R. Speicher, *Generalized statistics of macroscopic fields*, Lett.Math.Phys. **27** (1993), 97–104.
 - [TL] H.N.V. Temperley and E.H. Lieb, *Relations between the 'percolation' and 'colouring' problem, and other graph theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem*, Proc.Roy.Soc.Lond. **A322** (1971), 251–280.
 - [vDa] A. van Daele, *Quantum deformation of the Heisenberg group*, Current topics in operator algebras (H. Araki, ed.), World Scientific, Singapore, 1991, pp. 314–325.
 - [We1] R.F. Werner, *The Free quon gas suffers Gibbs' paradox*, Phys.Rev D **48** (1993), 2929–2934.
 - [We2] ———, *Finitely correlated pure states*, Proceedings of the workshop “On three levels”, Leuven 1993 (to appear).
 - [WZ] J. Wess and B. Zumino, *Covariant differential calculus on the quantum hyperplane*, Nuclear Physics B, Proceedings Supplements **18B** (1991), 302–312.
 - [Wo1] S.L. Woronowicz, *Twisted $SU(2)$ group. An example of a non-commutative differential calculus*, Publ.RIMS, Kyoto **23** (1987), 117–181.
 - [Wo2] ———, *Compact matrix pseudogroups*, Commun.Math.Phys. **111** (1987), 613–665.
 - [Wo3] ———, *Differential calculus on compact matrix pseudogroups (quantum groups)*, Commun.Math.Phys. **122** (1989), 125–170.
 - [Wo4] ———, *Unbounded elements affiliated with C^* -algebras and non-compact quantum groups*, Commun.Math.Phys. **136** (1991), 399–432.
 - [Zag] D. Zagier, *Realizability of a model in infinite statistics*, Commun.Math.Phys. **147** (1992), 199–210.

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